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NATURAL TRANSFORM AND SOLUTION OF INTEGRAL EQUATIONS FOR DISTRIBUTION SPACES

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Abstract

In this paper, integral equations such as Volterra convolution type of first and second type and Abel integral equation are solved using Nautral transform and further, the solutions so obtained are defined on certain distribution spaces.

Keywords : Natural transform, Fourier transform, Laplace transform, distribution spaces, integral equations

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1. Introduction

To solve differential and integral equations, several integral transforms (such as Fourier, Laplace, Sumudu and many more) are used [1, 2, 4, 5, 8, 10]. A new integral transform, the N - transform, is studied by Khan and Khan [9] and their properties and applications are described. The inverse of the transform and additional properties of the same are given by Belgacem et al. [3, 14]. Distributional Natural transform is defined by Loonker and Banerji [13].

This section of the paper deals with basic terminologies and properties of the Natural transform. In Section 2, the Natural transform is used to obtain solutions of integral equations. Section 3 deals with the application of the solution, so obtained in Section 2, on certain distribution spaces.

The real function $f(t) > 0$ and $f(t) = 0$ for $t < 0,$ is sectionwise continuous, of exponential order, and defined in the set A by

$$A = \{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < M e^{t/\tau_j}; t \in (-1)^j \times [0, \infty) \},$$

where M is a constant of finite number, τ_1 and τ_2 may be finite or infinite.

The Natural transform $R(s,u)$ of the function $f(t)$ for all $t \geq 0,$ is given by [3]

$$N^+[f(t)] = R(s,u) = \frac{1}{u}\int_0^\infty e^{-st} f(ut)dt \quad , \quad s > 0, u > 0 \tag{1}$$

i.e.

$$R(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)dt \tag{2}$$

where t, u are time variable and s is the frequency variable.

The discrete form of Natural transform is [3, p.102, Eqn. (2.14)]

$$N^+[f(t)] = R(s,u) = \sum_{n=0}^{\infty} \frac{n! a_n u^n}{s^{n+1}} \tag{3}$$

The inverse Natural transform is defined by [3, p.101, Eqn. (2.7)] and [14]

$$N^{-1}[R(s,u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s,u) ds \ . \tag{4}$$

The Natural transform is derived from the Fourier integral, see [3] where we notice also, duality relation between Natural-Laplace and Natural-Sumudu transform, together with other properties. We mention couple of properties of the Natural transform [3, 14].

1. *Natural transform of derivative* : The *derivative* of $f(t)$ with respect to t, and n *th order derivative* of $f(t)$ with respect to t are, respectively, defined by

$$N^+[f'(t)] = R_1(s,u) = \frac{s}{u} R(s,u) - \frac{f(0)}{u} \tag{5}$$

and

$$N[f^{(n)}(t)] = R_n(s,u) = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0) \ . \tag{6}$$

2. *Convolution Theorem* : If $F(s,u)$ and $G(s,u)$ are Natural transforms of the functions $f(t)$ and $g(t)$ defined in set A, then the *convolution* is given by

$$N^+[(f * g)(t)] = u\, F(s,u) G(s,u) \ . \tag{7}$$

3. When $f(t) = \delta(t)$ (the Dirac delta function), the Natural transform becomes

$$N^+[\delta(t)] = R(s,u) = \frac{1}{u}, \tag{8}$$

and when $f(t) = \frac{t^{n-1}}{\Gamma(n)}$ or $\frac{t^{n-1}}{(n-1)!}$; $n > 0$, the Natural transform is

$$N^+[\frac{t^{n-1}}{\Gamma(n)}] = \frac{u^{n-1}}{s^n}. \tag{9}$$

4. *Linearity property* : If α, β are any constants and $f(t)$ and $g(t)$ are functions, then

$$N^+[\alpha f(t) + \beta g(t)] = \alpha F(s,u) + \beta G(s,u) \ . \tag{10}$$

Integral equations occur in many field of mechanics and mathematical physics. They also arise as representation formulas for the solution of differential equations. Indeed, a differential equation can be replaced by an integral equation which incorporates its boundary conditions. Integral equations also form one of the most useful tools in many branches of pure analysis, see [6, 7, 10]. An integral equation in which an unknown function appears under one or more integral sign. The equations

$$f(s) = \int_a^b K(s,t) g(t) dt \quad , \tag{11}$$

$$g(s) = f(s) + \int_a^b K(s,t) g(t) dt \quad , \tag{12}$$

where the function $g(s)$ is the unknown function while all the other functions are known, are integral equations. These functions may be complex valued functions of real variables s and t. An integral equations (11) and (12) is called linear if only linear operational performed in it upon the unknown function. In fact, these can be written as

$$L[g(s)] = f(s) \tag{13}$$

where L is appropriate integral operator.
The most general type of linear integral equation is

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt \quad , \tag{14}$$

where the upper limit may be either variable or fixed. The functions f, h and K are known functions, where g is to be determined; λ is a non zero, real or complex parameter. The function $K(s,t)$ is called the kernel. The special cases of equations (14) are

(1) Fredholm integral equation : In this, the upper limit of integral b is fixed

(i) When $h(s) = 0$, then (14) will be

$$f(s) + \lambda \int_a^b K(s,t)g(t)dt = 0 \quad , \tag{15}$$

is known as Fredhlom integral equation of the first kind.

(ii) When $h(s) = 1$, then (14) becomes

$$f(s) + \lambda \int_a^b K(s,t)g(t)dt = 0 \quad , \tag{16}$$

is known as Fredhlom integral equation of the second kind.

(iii) When $f(s) = 0$ in (16),

$$g(s) = \lambda \int_a^b K(s,t)g(t)dt \quad , \tag{17}$$

called as homogenous Fredhlom integral equation.

(2) Volterra Equations : Volterra equations of the first, homogenous and second kinds are defined precisely as (15), (16), (17) expect that $b = s$ is the variable upper limit of integration.

(3) Convolution Integral Equation : The kernel $K(s,t)$ is considered as a function of the difference $(s-t)$ where k is a certain function of one variable. The integral equation

$$g(s) = f(s) + \lambda \int_a^b k(s-t)g(t)dt \quad , \tag{18}$$

is called as Fredlom equation of the convolution type.

(4) Abel Integral Equation is given as

$$f(s) = \int_a^s \frac{g(t)}{(s-t)^\alpha}, \quad s > a \tag{19}$$

and its solution is as follows

$$g(t) = \frac{\sin(\alpha\pi)}{\pi} \frac{d}{dt}\left[\int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}}\right] \tag{20}$$

2. Solution of Integral Equations and Natural Transform

In this section the Natural transform is invoked to obtain solutions of some integral equations and other applications can be seen in [6, 7, 10, 11, 12].

1. Consider the **Volterra integral equation of first kind with a convolution type kernel**

$$f(x) = \int_0^x k(s-t)g(t)dt \qquad , \tag{21}$$

where $k(s-t)$ depends only on the difference $(s-t)$. Applying the Natural transform to both the sides and using (7), we obtain

$$F(s,u) = u\,K(s,u)G(s,u)$$

i.e.

$$G(s,u) = \frac{F(s,u)}{u\,K(s,u)} \tag{22}$$

By using inversion Natural transform in (22), we obtain the solution of (21) as

$$g(x) = N^{-1}\left[\frac{F(s,u)}{u\,K(s,u)}\right] \tag{23}$$

2. Consider the **Volterra integral equation of second kind with a convolution type kernel**

$$g(x) = f(x) + \int_0^x k(s-t)g(t)dt \qquad , \tag{24}$$

On applying the Natural transform to both the sides and using convolution formula (7), (24) gives

$$G(s,\ u) = F(s,u) + u\,K(s,u)G(s,u)$$

i.e.

$$G(s,u) = \frac{F(s,u)}{1-u\,K(s,u)} \tag{25}$$

and the inverse Natural transform gives

$$g(x) = N^{-1}\left[\frac{F(s,u)}{1-u\,K(s,u)}\right] \tag{26}$$

which is the required solution of (24).

3. Consider the **Abel integral equation** in the form

$$f(t) = \int_0^x \frac{g(x)}{(t-x)^{\alpha}}dx \,, 0 < \alpha < 1 \tag{27}$$

which can be written as

$$f = g * t_+^{-\alpha} \qquad , \tag{28}$$

where $t_+^{-\alpha} = t^{-\alpha}H(t)$ and $t_+^{-\alpha}$ is Heaviside unit step function . Applying the Natural transform to both the sides and using the convolution (7) in (28), we have

$$N^+[f(t)] = u\,N^+[g(t)]\,N^+[t_+^{-\alpha}] \qquad . \tag{29}$$

When $f(t) = t^{\alpha-1}$, $N^+[t^{\alpha-1}] = \frac{\Gamma(\alpha)u^{\alpha-1}}{s^{\alpha}}$, and putting $\alpha = 1-\alpha$ in this or in (9), we obtain $N^+[t^{-\alpha}] = \frac{\Gamma(1-\alpha)u^{-\alpha}}{s^{(1-\alpha)}}$. Putting this value in (29), we get

$$N^+[f(t)] = u\,N^+[g(t)]\cdot\Gamma(1-\alpha)\frac{u^{-\alpha}}{s^{1-\alpha}} \qquad . \tag{30}$$

$$N^+[g(t)] = \frac{F(s,u)s^{1-\alpha}}{\Gamma(1-\alpha)u^{-\alpha+1}} \qquad .$$

$$= \frac{F(s,u)s^{(1-\alpha)}\Gamma(\alpha)}{u^{1-\alpha}\;\Gamma(\alpha)\Gamma(1-\alpha)}; \quad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

$$= \frac{\Gamma(\alpha)\sin(\pi\alpha)}{\pi}\,\frac{s^{(1-\alpha)}F(s,u)}{u^{(1-\alpha)}}$$

$$= \frac{\sin(\pi\alpha)}{\pi}\,\frac{\Gamma(\alpha)su^{\alpha-1}}{s^{\alpha}} \quad ; \; [\text{using (9)}]$$

i.e.

$$= \frac{\sin(\pi\alpha)}{\pi}\cdot F(s,u)\; s\; N^{+}[t^{\alpha-1}] \quad ; \; [\text{using (7)}]$$

$$= \frac{\sin(\pi\alpha)}{\pi}\,\frac{s}{u}\,N^{+}[t^{\alpha-1} * f(t)] \;\;,$$

$$= \frac{\sin(\pi\alpha)}{\pi}\,\frac{s}{u}\,N^{+}\left[\int_0^t (t-x)^{\alpha-1} f(x)dx\right]$$

i.e.

$$N^{+}[g(t)] = \frac{\sin(\pi\alpha)}{\pi}\,\frac{s}{u}\,N^{+}[K(t)] \quad , \tag{31}$$

where $K(t) = \int_0^t (t-x)^{\alpha-1} f(x)dx, K(0) = 0$.

From (5), $N^{+}[K'(t)] = \frac{s}{u}R(s,u) = \frac{s}{u}N^{+}[K(t)],$ which when invoked in (31), we get

$$N^{+}[g(t)] = \frac{\sin(\pi\alpha)}{\pi}\,\frac{s}{u}\,\frac{u}{s}\,N^{+}[K'(t)]$$

i.e. $$N^{+}[g(t)] = \frac{\sin(\pi\alpha)}{\pi}\,N^{+}\left[\frac{dK(t)}{dt}\right]$$

Therefore, the complete solution of (27) is given by

$$g(t) = \frac{\sin(\pi\alpha)}{\pi}\,\frac{d}{dt}\left[\int_0^t (t-x)^{\alpha-1} f(x)dx\right] \quad . \tag{32}$$

In what follows are some illustrative examples which support the use of the Natural transform to solve integral equations.

Example 1 : Find the function $g(x)$ which satisfies the equation

$$g(x) = x + \int_0^x g(t)\sin(x-t)dt \tag{33}$$

Solution : Invoking the Natural transform on both the sides and (7) in (33), we obtain

$$G(s,u) = \frac{u}{s^2} + uG(s,u)\frac{u}{(s^2+u^2)}$$

i.e. $$s^2(s^2+u^2)G(s,u) = u(u^2+s^2) + u^2s^2G(s,u)$$

i.e. $$(s^4+u^2s^2)G(s,u) - u^2s^2G(s,u) = u(u^2+s^2)$$

i.e. $$G(s,u)=\frac{u}{s^2}+\frac{u^3}{s^4}$$

Taking inverse Natural transform, we get

$$g(x)=x+\frac{x^3}{3!} \quad , \tag{34}$$

which is the required solution.

Example 2 : Solve

$$g(x)=1+\lambda\int_0^x (t-x)g(t)dt \ , (\lambda=\text{constant}) \tag{35}$$

Solution : Using the Natural transform on both the sides, we have

$$G(s,u)=\frac{1}{s}+\lambda u G(s,u)\left(-\frac{u}{s^2}\right)$$

i.e.

$$s^2 G(s,u)=s-\lambda u^2 G(s,u)$$

$$G(s,u)(s^2+\lambda u^2)=s$$

i.e.

$$G(s,u)=\frac{s}{s^2+\lambda u^2}$$

Taking inverse Natural transform of the above equation, we get

$$g(x)=\cos(x\sqrt{\lambda}) \tag{36}$$

which is the required solution.

3. Distributional Solution of Integral Equations

In this section we explain that the solution obtained in Section 2 of the integral equations, can be defined on the distribution spaces. In order to define integral equations on distribution spaces, we need to specify the space (or spaces) of generalized function. Then, we need to give an *interpretation of the equation in terms of an operator defined in that space of distributions*. This interpretation should be such that when applied to ordinary functions, integral equation can be recovered. *One is the space* $\mathcal{D}'_{41}[a,\infty)$, which is known as mixed distribution space [cf. [6]] that can be identified with the space of distribution $\mathcal{D}'(R)$ whose support is $[a,\infty)$. Another distribution space is $\mathcal{D}'_{43}[a,\infty)$, which can be identified with the space $S'(R)$ (tempered distribution space) whose support is contained in $[a,\infty)$.

The interpretation of integral equation can be achieved by using the concept of convolution of distributions. If both u and v have supports bounded on the left, then $u*v$ is always defined. Actually, if supp $u\subseteq[a,\infty)$ and supp $v\subseteq[b,\infty)$, then supp $u*v\subseteq[a+b,\infty)$. Thus, the convolution can be considered as a bilinear operation $*:\ \mathcal{D}'_{41}[a,\infty)\times\mathcal{D}'_{41}[b,\infty)\to\mathcal{D}'_{41}[a+b,\infty)$. If $u\in\mathcal{D}'_{41}[a,\infty)$ and $v\in\mathcal{D}'_{41}[b,\infty)$ are locally integrable functions, then we have

$$(u*v)(t)=\int_a^{t-b} u(\tau)v(t-\tau)d\tau \quad , \quad t>a+b \quad . \tag{37}$$

When $b=0$ and $v\in\mathcal{D}'_{41}[0,\infty)$, we have $u*v\in\mathcal{D}'_{41}[a,\infty)$. Thus the convolution, with v, defines an operator of the space $\mathcal{D}'_{41}[a,\infty)$, which is given by

$$(u * v)(t) = \int_a^t u(\tau)v(t-\tau)d\tau \quad , \quad t > a \quad , \tag{38}$$

where u and v are locally integrable functions. *The integral equation and its solution can be interpreted in the distributional sense*.

Moreover, *we define the integral transform on distributional spaces and further using the distributional integral transform to obtain the solution of integral equation,* can be interpreted as distributional sense. For this one may refer [10, 11,12, 13]. The Natural transform $R(s,u)$ of the function $f(t) \in$ ☟′ can be written as [13]

$$N^{+}[f(t)] = R(s,u) = \left\langle f(t), \frac{1}{u} e^{-\frac{st}{u}} \right\rangle \ . \tag{39}$$

Few properties such as convolution, Parseval equation of Natural transform on distribution spaces is also defined.

Using the concept of convolution of distributions for integral equations and distributional Natural transform as defined above, we can conclude that the solution obtained in Section 2 for the integral equations using Natural transform can be considered in the distribution sense. The known and the unknown functions of the integral equations solved, using distributional Natural transform, are characterized for the spaces of distribution such as ☟′, ☟′$_{41}$, ☟′$_{43}$ and any other distribution spaces. In similar manner other forms of integral equations can be solved using Natural transform and can also be invoked for distribution spaces.

Conclusion : The integral transform, namely Natural transform (abbreviated as N transform), is used to solve Volterra convolution type of first kind and second type and Abel integral equations. Few illustrations are proposed to clarify the analysis incorporated in the present paper. The distribution spaces are defined for the solution of integral equations as obtained using classical and distributional Natural transform.

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