American J. of Mathematics and Sciences
Vol. 3, No -1 ,(January 2014)

ISSN No: 2250-3102

A Lower Bound Theorem

Lin Hu

Department of Applied Mathematics, Beijing University of Technology, Beijing 100124, P. R. China

Department of Basic Courses, Beijing Union University, Beijing 100101, P. R. China

e-mail: hulin9803@yeah.net

Abstract Motivated by Candes and Donoho′s work (Candés, E J, Donoho, D L, Recovering edges in ill-posed inverse problems: optimality of curvelet frames. Ann. Stat. 30, 784-842 (2002)), this paper is devoted to giving a lower bound of minimax mean square errors for Riesz fractional integration transforms and Bessel transforms.

Keywords Bessel transform; Riesz transform; error; noise.

AMS(2000) subject classifications 42C15, 42C40

1. Introduction

The linear inverse problem for a statistical model with additive noise plays important roles in scientific settings ranging from medical imaging to physical chemistry ([1]). More precisely, consider the problem of recovering an image f from the noisy data

$$Y = Kf + \varepsilon W. \tag{1.1}$$

Here, f belongs to $\varepsilon^2(A)$, which is the function space consisting of compactly supported and twice continuously differentiable away from a smooth edge; W denotes a Wiener sheet; ε is a noisy level; K stands for a linear operator from $L^2(\mathbb{R}^2)$ to another Hilbert space. We use $\underset{V}{sup}\, E\|\hat{f}-f\|_2^2$ to denote the mean square error on the function space V for the L^2 risk and $\mathcal{M}(\varepsilon, V) := \underset{\hat{f}}{inf}\ \underset{V}{sup}\, E\|\hat{f}-f\|_2^2$ to represent the minimax mean square error on the function space V for the L^2 risk. In 2010, Colonna and Easley use shearlets to deal with the inverse problem (1.1), when K is the Radon transform([3]). They give an upper bound of the mean square error $\varepsilon^{\frac{4}{5}}(log(\varepsilon^{-1}))$ on the space $\varepsilon^2(A)$ for the L^2 risk. Moreover, they show a lower bound to the minimax mean square error $\varepsilon^{\frac{4}{5}}(log(\varepsilon^{-1}))^{-\frac{2}{5}}$ for that class of functions, which means their upper bound essentially optimal, ignoring log factor.

Note that Riesz fractional integration transforms and Bessel transforms play important roles in both theoretical analysis and practical applications. Hu and Liu ([5])apply shearlets to the inverse problem (1.1) for a family of linear operators including Riesz fractional integration transforms and Bessel transforms. Based on a shearlet shrinkage method, they obtain an upper bound of the mean square error $\varepsilon^{\frac{2}{3/2+2\alpha}}(log(\varepsilon^{-1}))$ on the space $\varepsilon^2(A)$ for the L^2 risk. The goal of this paper is to give a lower bound of the minimax mean square error for Riesz fractional integration transforms and Bessel transforms. It turns out that the above mentioned upper bounds are optimal, ignoring log factor.

2. Main Theorem

To introduce our main theorem, we begin with the definitions of Riesz fractional integration transforms and Bessel transforms ([6]).

Definition 1 Riesz fractional integration transform I_α is defined by

$$I_\alpha(f)(x) = C_\alpha \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{2-\alpha}} dy \qquad (0<\alpha<\tfrac{1}{2})$$

with some normalizing constant C_α ; the Bessel operator B_α by

$$B_\alpha f = G_\alpha * f \qquad (0<\alpha<2)$$

with $G_\alpha(x) = A_\alpha \dfrac{K_{1-\frac{\alpha}{2}}(|x|)}{|x|^{1-\frac{\alpha}{2}}} \in L^1(\mathbb{R})$, where A_α is a normalizing constant and

$K_v(z)$ represents McDonald function defined as

$$K_v(z) = \frac{1}{2}(\frac{z}{2})^{-v} \int_0^\infty t^{v-1} e^{-t-\frac{z^2}{4t}} dt.$$

The following two lemmas play an important role in our discussion.

Lemma 1 ([7]) Let $f \in L^p(\mathbb{R}^2)$ and I_α be the Riesz fractional integration transform. If $1 < p < \frac{2}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}$, then there exist a constant $C > 0$ such that

$$\|I_\alpha f\|_q \leq C\|f\|_p.$$

Lemma 2 ([2, 4]) For $N \geq 1$, let $\xi \in \{0,1\}^N$ and $X \sim N(\xi, V)$ be a multivariate Gaussian vector. Assume that V is invertible such that $\tau_i^2 = Var(X_i | X_k, k \neq \neq i) = \frac{1}{(V^{-1})_{ii}} \geq 1$ for all $1 \leq i \leq N$. Then there is an absolute constant B such that

$$\inf_{\hat{\xi}} \sup_{\xi \in \{0,1\}^N} E\|\hat{\xi} - \xi\|_2^2 \geq BN.$$

Now, we are ready to state the main theorem of this paper:

Theorem If the operator K in (1.1) take I_α or B_α, then there exits a constant $C > 0$ such that

$$\mathcal{M}(\varepsilon, \varepsilon^2(A)) \geq C\varepsilon^{\frac{2}{3/2+2\alpha}} \quad (\varepsilon \to 0).$$

Proof. Firstly, we consider Riesz fractional integration transform I_α: Let h be a smooth function of variable t with compact support contained in $[0, 2\pi]$ and

$$h_{m,j}(t) = m^{-\frac{2\alpha+2}{2\alpha+1}} h(mt - 2\pi j), \quad j = 0, 1, \cdots, m-1$$

for $m \geq 1$. We introduce a polar coordinates (r, θ) with origin at $(\frac{1}{2}, \frac{1}{2})$. Set $r_0 = \frac{1}{4}$, $f_0 := 1_{\{r \leq r_0\}}$, where 1_A denotes the indicator function on a set A. Then the functions

$$\psi_{m,j} := 1_{\{r < h_{m,j} + r_0\}} - f_0, \quad j = 0, 1, \cdots, m-1$$

are disjointly supported and $\|\psi_{m,j}\|_2^2 = \|h_{m,j}\|_1 = m^{-\frac{4\alpha+3}{2\alpha+1}} \|h\|_1 \sim m^{-\frac{4\alpha+3}{2\alpha+1}}$. Here and after, $A \sim B$ denotes $A \leq CB$ and $A \geq CB$ for some constant $C > 0$).

Let

$$\mathcal{H}_m := \{f = f_0 + \sum_{j=0}^{m-1} \xi_j \psi_{m,j}\}.$$

Then $\mathcal{H}_m \subseteq \varepsilon^2(A)$, which implies

$$\mathcal{M}(\varepsilon, \varepsilon^2(A)) \geq \mathcal{M}(\varepsilon, \mathcal{H}_m) := \underset{\hat{f}}{inf} \underset{\mathcal{H}_m}{sup} E\|\hat{f} - f\|_2^2. \tag{2.1}$$

So, the problem reduces to estimate $f \in \mathcal{H}_m$. Furthermore, we can restrict the estimator of the form:

$$\hat{f} = f_0 + \sum_{j=0}^{m-1} \hat{\xi}_j \psi_{m,j}.$$

In fact, let P_m denote the L^2 projection on the smallest affine subspace containing $\mathcal{H}_m$. Then for $f \in \mathcal{H}_m$,

$$\|P_m \hat{f} - f\|_2^2 = \|P_m \hat{f} - P_m f\|_2^2 \leq \|\hat{f} - f\|_2^2.$$

This implies the risk of a general estimator $\hat{f}$ greater than or equal to that of a corresponding estimator $P_m \hat{f}$. Moreover,

$$\|\hat{f} - f\|_2^2 = \|\sum_{j=0}^{m-1} (\hat{\xi}_j - \xi_j)\psi_{m,j}\|_2^2 \sim \|\hat{\xi} - \xi\|_2^2 \|\psi_{m,j}\|_2^2 \sim m^{-\frac{4\alpha+3}{2\alpha+1}} \|\hat{\xi} - \xi\|_2^2 \tag{2.2}$$

due to the orthogonality of $\psi_{m,j}$ and the fact that $\|\psi_{m,j}\|_2^2 \sim m^{-\frac{4\alpha+3}{2\alpha+1}}$. Hence, it is sufficient to estimate $\xi \in \{0,1\}^M$.

Let $g_j := I_\alpha \psi_{m,j}$. Applying Lemma 1 to g_j with $q = 2$, $p = \dfrac{2}{1+\alpha}$, one has $\|g_j\|_2^2 = \|I_\alpha \psi_{m,j}\|_2^2 \leq \|\psi_{m,j}\|_{\frac{2}{1+\alpha}}^2$. This with the fact $\|\psi_{m,j}\|_{\frac{2}{1+\alpha}}^2 = \|h_{m,j}\|_1^{1+\alpha} \leq$ $\leq m^{-\frac{(4\alpha+3)(\alpha+1)}{2\alpha+1}}$ leads to

$$\|g_j\|_2^2 \leq m^{-\frac{(4\alpha+3)(\alpha+1)}{2\alpha+1}}.$$

Because a Riesz fractional integration transform I_α is invertible, the functions g_j are linearly independent. Let V_m stand for the smallest affine space containing $I_\alpha f_0 + \sum\limits_{j=0}^{m-1} \theta_j g_j$ for arbitrary $\{\theta_j\}_{j=0}^{m-1} \in \{0,1\}^m$. Note that, for each function $v(x) \in L^2(\mathbb{R}^2)$ orthogonal to V_m, the law of $\int_{\mathbb{R}^2} v(x) Y dx$ is $N(0, \|v\|^2)$ independently of ξ. So, the projection of the Riesz fractional integration data on the span V_m is sufficient for ξ.

Since $\{g_j\}_{j=0}^{m-1}$ is linear independent, the linear functions $\langle g_j, f - Rf_0\rangle$ give a nondegenerate set of affine coordinates for $f \in V_m$. For each j, define

$$Y_j := \langle Y, g_j\rangle - \langle I_\alpha f_0, g_j\rangle = \sum_{i=0}^{m-1}\langle g_j, g_i\rangle\xi_i + \sum_{i=0}^{m-1}\varepsilon\langle W, g_i\rangle.$$

Then the vector $Y := (Y_j)_{j=0}^{m-1}$ gives a nondegenerate set of affine coordinates for the projection of the Riesz fractional integration data on the space V_m. Hence, $Y = (Y_j)_{j=0}^{m}$ is a sufficient statistic for ξ and $Y \sim N(G\xi, \varepsilon^2 G)$, where G is the matrix with the i, j element $G_{j,i} = \langle g_j, g_i\rangle$.

Because g_j is linearly independent, the matrix G is invertible. Define $X :=$ $= G^{-1}Y$, then $X \sim N(\xi, \varepsilon^2 G^{-1})$. Note that Y is a sufficient statistic for ξ, so is X. We may restrict our attention to estimator X. Let $V := \varepsilon^2 G^{-1}$ be the covariance matrix of X and

$$\tau_j^2 := Var(X_j|X_k, k \neq j)$$

be the conditional variance of X_i given the other coordinates. Take m such that $m \sim \varepsilon^{-\frac{2\alpha+1}{(\frac{3}{2}+2\alpha)(\alpha+1)}}$. Then $\|g_j\|^2 \leq m^{-\frac{(4\alpha+3)(\alpha+1)}{2\alpha+1}} \leq \varepsilon^2$ for all $1 \leq j \leq m-1$ and $\tau_j^2 = Var(X_j|X_k, k \neq j) = \dfrac{1}{(V^{-1})_{jj}} = \dfrac{1}{\varepsilon^{-2}(G)_{jj}} = \varepsilon^2\|g_j\|^{-2} \geq 1$. By Lemma 2,

$$\inf_{\hat{\xi}}\ \sup_{\xi\in\{0,1\}^m} E\|\hat{\xi}_j - \xi_j\|_2^2 \geq Bm.$$

This with (2.1) and (2.2) shows

$$\mathcal{M}(\varepsilon, \varepsilon^2(A)) \geq \inf_{\hat{f}}\sup_{\mathcal{H}_m} E\|\hat{f} - f\|_2^2 \geq Bm^{-\frac{2\alpha+2}{2\alpha+1}}.$$

Using $m \sim \varepsilon^{-\frac{2\alpha+1}{(\frac{3}{2}+2\alpha)(\alpha+1)}}$. one receives $\mathcal{M}(\varepsilon, \varepsilon^2(A)) \geq C\varepsilon^{\frac{2}{3/2+2\alpha}}$. This completes the proof for $K = I_\alpha$.

It remains to conclude the theorem for $K = B_\alpha$. By Definition 1, one has $G_\alpha(x) = \dfrac{A_\alpha}{2^{\frac{\alpha}{2}}}\dfrac{1}{|x|^{2-\alpha}}\int_0^\infty t^{-\frac{\alpha}{2}}e^{-t-\frac{x^2}{4t}}dt \leq \dfrac{C}{|x|^{2-\alpha}}$. Hence,

$$\|B_\alpha\psi_{m,j}\|^2 = \|B_\alpha * \psi_{m,j}\|^2 \leq \|I_\alpha\psi_{m,j}\|^2 \leq m^{-\frac{(4\alpha+3)(\alpha+1)}{2\alpha+1}}$$

due to $\psi_{m,j} > 0$. Note that the Bessel operator B_α is invertible. Then the exactly same arguments as above show $\mathcal{M}(\varepsilon, \varepsilon^2(A)) \geq C\varepsilon^{\frac{2}{3/2+2\alpha}}$ for Bessel transform B_α. This completes the proof of our main theorem.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11271038) and Natural Science Foundation of Beijing (No. 1082003). The author would like to thank her adviser, Prof. Youming Liu for his important suggestions and comments.

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