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Semi-invariant Submanifolds of a General p-contact Hsu-metric Manifold

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Abstract

In 1972, Vanzura [5] defined and studied almost r-contact structure which is the generalization of (f, g, u, v, λ)-structure manifold introduced by Yano and Okumura [6]. Some properties of almost r-contact structure manifolds have also been studied by Nivas and Singh [4]. CR-structure manifolds have been defined and studied by Bejancu ([1], [2]) and many other geometers. In the present paper, semi- invariant submanifolds of general almost p-contact Hsu-metric manifold have been studied. The idea of CR-structure is extended and certain interesting results have been obtained.

Keywords and Phrases: Semi-invariant submanifolds, p-contact Hsu-metric structure, Distributions, Integrability.

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1. Preliminaries:

Let M^{n+p} be an $(n+p)$-dimensional differentiable manifold of class C^∞. Suppose there exists on M^{n+p} a tensor field ϕ of type (1, 1), $p(C^\infty)$ contravariant vector fields $\xi_1, \xi_2, ..., \xi_p$ and $p(C^\infty)$ 1-forms $\eta^1, \eta^2, ..., \eta^p$ (p some finite integer) satisfying

(1.1) $$\phi^2 = a^r I + \sum_{l=1}^{p} \eta^l \otimes \xi_l$$

where $'a'$ is any real or complex number not zero and $'r'$ a positive integer.

Also

(1.2)
$$(i) \quad \phi\xi_l = 0$$
$$(ii) \quad \eta^l \circ \phi = 0$$
$$(iii) \quad \eta^l(\xi_m) + a^r \delta^l_m = 0$$

where $l, m = 1,2,......, p$ and δ^l_m denotes the Kronecker delta. Thus the manifold M^{n+p} in view of the equations (1.1) and (1.2) will be said to possess the general p-contact Hsu structure. Suppose further that the above manifold M^{n+p} admits a positive definite Riemannian metric g satisfying

(1.3) $$g(\phi X, \phi Y) + a^r g(X, Y) + \sum_{l=1}^{p} \eta^l(X)\eta^l(Y) = 0$$

and consequently

(1.4) $$g(\xi_l, Y) = \eta^l(Y), \quad l = 1,2,......., p$$

Then the above manifold M^{n+p} will be called the general p-contact Hsu-metric structure manifold [3].

Let V^n be an n-dimensional differentiable manifold immersed differentiably in M^{n+p}. Let us assume that the vector fields ξ_l are tangents to V^n. We say that V^n is semi-invariant submanifold of M^{n+p} if there exist differentiable distributions D and $D^\perp$ on V^n such that

$(i) \quad T(V^n) = \{D\} + \{D^\perp\} + \{\xi_l\}$

$\{D\},\{D^{\perp}\}$ and $\{\xi_l\}$ are mutually orthogonal and $T(V^n)$ is tangent bundle of V^n.

(ii) The distribution D is invariant by ϕ i.e.

$$\phi(D_x) = D_x \quad , \quad \text{for all } x \text{ in } V^n$$

(iii) The distribution $\{D^{\perp}\}$ is anti invariant by ϕ i.e.

$$\phi(D_x^{\perp}) \subset T_x^{\perp}(V^n)$$

where $T_x^{\perp}(V^n)$ denotes the normal space of V^n at $x \in V^n$ ([1], [2])

Suppose further that $\overline{\nabla}$ and ∇ are Levi-civita connections on M^{n+p} and V^n respectively. Then Gauss and Weingarten equations can be expressed as [2]

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.5}$$
$$and$$
$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V$$

where X,Y are arbitrary tangent vectors and V the vector field normal to V^n. Also $h(X,Y)$ is second fundamental form and A_V the shape operator given by

$$g(A_V X,Y) = g(h(X,Y),V) \tag{1.6}$$

Let us denote by P,Q the projection morphisms of $T(V^n)$ on D and $D^{\perp}$ respectively. Then we can write

$$X = PX + QX + \sum_{l=1}^{p} \eta^l(X)\xi_l \tag{1.7}$$

for any vector field X tangent to V^n. We can also write

$$\phi V = BV + CV \tag{1.8}$$

for V normal to V^n. We can also define maps $\psi : T(V^n) \to T(V^n)$ and $\omega : T(V^n) \to T(V^n)^{\perp}$ as follows

$$(i) \quad \psi X = \phi PX \tag{1.9}$$
$$(ii) \quad \omega X = \phi QX$$

Thus in view of the equations (1.2), (1.7) and (1.9), it follows that

$$\phi X = \psi X + \omega X \tag{1.10}$$

Here ψX denotes the tangential part and ωX the normal part of ϕX.
We can also write
Operating (1.10) by ϕ and making use of the equations (1.1), (1.8) and (1.10), we get

$$a^r X + \sum_{l=1}^{p} \eta^l(X)\xi_l = \psi^2 X + \omega\psi X + B\omega X + C\omega X$$

Comparison of tangential and normal vectors to V^n gives

$$(i) \quad \psi^2 = a^r I_n + \sum_{l=1}^{p} \eta^l \otimes \xi_l - B\omega \tag{1.11}$$
$$(ii) \quad \omega\psi + C\omega = 0$$

In a similar manner, premultiplying the equation (1.8) by ϕ and making use of (1.1), (1.8) and (1.10), we get

$$(i) \quad C^2 = a^r - \omega B \tag{1.12}$$
$$(ii) \quad \psi B + BC = 0$$

Hence we have

Theorem: Let M^{n+p} be an $(n+p)$-dimensional differentiable manifold admitting the general p-contact Hsu structure and V^n be the semi-invariant manifold immersed differentiably in M^{n+p}. Then the structure induced on the submanifold is given by the equations (1.11) and (1.12).

2. Parallel Fields:

Let us now suppose that for the enveloping manifold M^{n+p}, the tensor field ϕ is parallel. Thus

$$(\overline{\nabla}_X \phi)Y = 0$$

or

(2.1) $$\overline{\nabla}_X \phi Y = \phi \overline{\nabla}_X Y$$

In view of the equations (1.8) and (1.10), above equation takes the form

$$\overline{\nabla}_X \{\psi Y + \omega Y\} = \phi\{\nabla_X Y + h(X,Y)\}$$

or

(2.2) $$\nabla_X \psi Y + h(X,\psi Y) - A_{\omega Y} X + \nabla_X \omega Y = \psi(\nabla_X Y) + \omega(\nabla_X Y) + Bh(X,Y) + Ch(X,Y)$$

Comparison of tangential and normal fields in the above equation (2.2) yields

(2.3) $$(i) \quad (\nabla_X \psi)Y = A_{\omega Y} X + Bh(X,Y)$$
$$(ii) \quad (\nabla_X \omega)Y = Ch(X,Y) - h(X,\psi Y)$$

Further since ϕ is parallel on the enveloping manifold, we have

(2.4) $$(\overline{\nabla}_X \phi)V = 0$$

Proceeding in a way similar to above and using (1.5), (1.6) and (1.8), we get

(2.5) $$(i) \quad (\nabla_X B)V = A_{CV} X - \psi(A_V X)$$
$$(ii) \quad (\nabla_X C)V + h(X, BV) + \omega A_V X = 0$$

3. Integrability Conditions:

In this section, we shall study integrability of distributions D and $D^{\perp}$. If $N_\phi(X,Y)$ and $N_\psi(X,Y)$ be the Nijenhuis tensors of ϕ and ψ respectively, we have in view of the equations (1.9), (1.10) and the definition of Nijenhuis tensor

(3.1) $$N_\phi(X,Y) = N_\psi(X,Y) - \omega\{[\phi X, Y] + [X, \phi Y] - \psi[X,Y]\} + B\omega([X,Y]) + C\omega([X,Y])$$

for all vector fields X, Y in D.

For all vector fields Z on the enveloping manifold, we denote by tZ the tangential part of Z and $t^{\perp}Z$ by normal part of it. In view of the equation (3.1), we have

Theorem 3.1: The necessary and sufficient condition for the distribution D to be integrable is that

(3.2) $$(i) \quad tN_\phi(X,Y) = N_\psi(X,Y)$$
$$and$$
$$(ii) \quad t^{\perp}N_\phi(X,Y) = -\omega\{[\phi X, Y] + [X, \phi Y] - \psi[X,Y]\}$$

for all X, Y in D.

Theorem 3.2: The necessary and sufficient condition for the distribution D to be integrable is that

(3.3) $$(i) \quad t^{\perp}N_\phi(X,Y) = 0$$
$$and$$
$$(ii) \quad QN_\psi(X,Y) = 0$$

for all X, Y in D.

Proof: If the distribution D is integrable, (3.3) (i) follows from (3.2)(ii). For the Nijenhuis tensor $N_\psi(X,Y)$, we have

$$N_\psi(X,Y) = [\psi X, \psi Y] - \psi[\psi X, Y] - \psi[X, \psi Y] + \psi^2[X,Y]$$

for all X, Y in D.

Thus $N_\psi(X,Y) \in D$ and consequently $QN_\psi(X,Y) = 0$.

If $X, Y \in D^\perp$, we can show

$$N_\psi(X,Y) = -P([X,Y]).$$

Thus we have

Theorem 3.3: The necessary and sufficient condition for the distribution $D^\perp$ to be integrable is that

$$N_\psi(X,Y) = 0 \text{, for all } X, Y \text{ in } D^\perp.$$

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