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Some sufficient conditions of strlikeness for the class of non-Bazilevič functions

Lifeng Guo[1*] Yi Ling[2] Gejun Bao[3]

1. School of Mathematica Science and Technology, Northeast Petroleum University, Daqing 163318, China.
2. Department of Mathematics Delaware State University Dover, DE 19901, U. S. A.
3. Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China.

Abstract

In this paper we consider a class of analytic and multivalent functions to obtain some sufficient conditions of strlikeness for the class of non-Bazilevič functions.

Keywords: Non-Bazilevič functions; Multivalent analytic functions; Multivalent starlike functions.

Mathematics Subject Classification: 30C45

1 Introduction

Let $H_p(n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, (n, p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f(z) \in H_p(n)$ is said to be in the subclass $S^*(p, n, \alpha)$ of multivalent starlike functions of order α in U if it satisfies the following inequality

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha, (z \in U, 0 \leq \alpha < p, p \in \mathbb{N}), \tag{1.2}$$

Obradovic ([1]) introduced a class of functions $f(z) \in H$ such that, for $0 < \alpha < 1$,

$$\text{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > 0, z \in U.$$

*Corresponding author. E-mail address: hitglf@yahoo.com.cn.

He called this class of function as "non-Bazilevič" type. Tuneski and Darus ([2]) obtained the Fekete-Szegö inequality for the non-Bazilevič class of functions.

Definition 1.1. Let $\alpha \geq 0, 0 \leq \beta < p$, a function $f(z) \in H_p(n)$ is in the class $\mathcal{N}(p,n,\alpha,\beta,g(z))$, if there exists a function g belonging to the class $S^*(p,n) := S^*(p,n,0)$ such that

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p\right| < p-\beta, (z \in U, p \in \mathbb{N}). \tag{1.3}$$

In particular, when $\alpha = 0$, a function $f \in \mathcal{N}(p,n,0,\beta) := S^*(p,n,\beta)$.

In the present paper, if $f(z) \in H_p(n)$ satisfy anyone of the certain inequalities, we obtain the $f(z) \in \mathcal{N}(p,n,\alpha,\beta)$.

To prove our main result, we need the following Lemma:

Lemma 1.1.1. ([3]). Let the function $w(z)$ be(nonconstant) analytic in U with $w(0) = 0$. If $|w(z)|$ attsts its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then

$$z_0 w'(z_0) = kw(z_0), \tag{1.4}$$

where k real and $k \geq 1$.

2 Main Results

Our main result is the following:

Theorem 2.1. Let $z \in U, \alpha \geq 0, 0 \leq \beta < p, p \in \mathbb{N}$. Suppose that $f(z) \in H_p(n)$, and $g(z) \in S^*(p,n)$ satisfy anyone of the following inequalities:

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1\right)\right| < p-\beta, \tag{2.1}$$

$$\left|\frac{\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}}\right| < \frac{p-\beta}{(2p-\beta)^2}, \tag{2.2}$$

$$\left|\frac{\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p}\right| < \frac{1}{2p-\beta}, \tag{2.3}$$

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\frac{\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p}\right| < 1, \tag{2.4}$$

$$\begin{aligned}\left|\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1\right. & \left. +\delta\left(\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p\right)\right| \\ & < \frac{(p-\beta)[\delta(2p-\beta)+1]}{2p-\beta}.\end{aligned} \tag{2.5}$$

then $f(z) \in \mathcal{N}(p, n, \alpha, \beta, g(z))$
Proof. Let $f(z) \in H_p(n)$ and $g(z) \in S^*(p, n)$. Define a function $w(z)$ such that

$$\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} = p + (p-\beta)w(z), (z \in U, \alpha \geq 0, 0 \leq \beta < p, p \in \mathbb{N}) \tag{2.6}$$

Here $w(z)$ is analytic in U with $w(0) = 0$. Then it follows from the above definition (2.6) that

$$\left(\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1\right) = \frac{(p-\beta)zw'(z)}{p+(p-\beta)w(z)} \tag{2.7}$$

Hence, from (2.6) and (2.7), we have

$$F_1(z) = \frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1\right) = (p-\beta)zw'(z), \tag{2.8}$$

$$F_2(z) = \frac{\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}} = \frac{(p-\beta)zw'(z)}{(p+(p-\beta)w(z))^2}, \tag{2.9}$$

$$F_3(z) = \frac{\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p} = \frac{zw'(z)}{w(z)}\frac{1}{p+(p-\beta)w(z)}, \tag{2.10}$$

$$F_4(z) = \frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\frac{\frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1}{\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p} = \frac{zw'(z)}{w(z)}, \tag{2.11}$$

$$\begin{aligned} F_5(z) \quad &= \frac{zf''(z)}{f'(z)} - (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zg'(z)}{g(z)} + 1 + \delta\left(\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p\right) \\ &= \delta(p-\beta)w(z) + \frac{(p-\beta)zw'(z)}{p+(p-\beta)w(z)}. \end{aligned} \tag{2.12}$$

Now from Lemma 1.1, suppose that there exist $z_0 \in U$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1.$$

Therefore letting $w(z_0) = e^{i\theta}$ in each of (2.8)-(2.12),we obtain that

$$|F_1(z_0)| = |(p-\beta)zw'(z_0)| = |(p-\beta)ke^{i\theta}| \geq p-\beta, \tag{2.13}$$

$$|F_2(z_0)| = \left|\frac{(p-\beta)zw'(z_0)}{(p+(p-\beta)w(z_0))^2}\right| = \frac{|(p-\beta)ke^{i\theta}|}{|(p+(p-\beta)e^{i\theta})^2|} \geq \frac{p-\beta}{(2p-\beta)^2}, \tag{2.14}$$

$$|F_3(z_0)| = \left|\left(\frac{zw'(z_0)}{w(z_0)}\right)\frac{1}{p+(p-\beta)w(z_0)}\right| = \left|\frac{|ke^{i\theta}|}{e^{i\theta}[p+(p-\beta)e^{i\theta}]}\right| \geq \frac{1}{2p-\beta}, \tag{2.15}$$

$$|F_4(z_0)| = \left|\frac{zw'(z_0)}{w(z_0)}\right| = |k| \geq 1, \tag{2.16}$$

$$|F_5(z_0)| = \left|\delta(p-\beta)w(z_0) + \tfrac{(p-\beta)zw'(z_0)}{p+(p-\beta)w(z_0)}\right| \begin{array}{l} = \left|\delta(p-\beta)e^{i\theta} + \tfrac{k(p-\beta)e^{i\theta}}{p+(p-\beta)e^{i\theta}}\right| \\ \geq \tfrac{(p-\beta)(\delta(2p-\beta)+1)}{2p-\beta}, \end{array} \tag{2.17}$$

which contradicts our assumption (2.13)-(2.17), respectively. Therefore $|w(z)| < 1$ hold true for all $z \in U$. Thus from (2.6) we have

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} - p\right| = (p-\beta)|w(z)| < p-\beta, (z \in U),$$

which implies that $f(z) \in \mathcal{N}(p, n, \alpha, \beta, g(z))$.

The Theorem 2.1 yields many interesting and important consequences. Some of these are given here. First of all, on setting $\alpha = 0$,in Theorem 2.1 ,we get

Corollary 2.1. Let $z \in U, 0 \leq \beta < p, p \in \mathbb{N}$. Suppose that $f(z) \in H_p(n)$ satisfy anyone of the following inequalities:

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right)\right| < p-\beta, \tag{2.18}$$

$$\left|\frac{f(z)}{zf'(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z)} + 1\right)\right| < \frac{p-\beta}{(2p-\beta)^2}, \tag{2.19}$$

$$\left|\frac{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1}{\frac{zf'(z)}{f(z)} - p}\right| < \frac{1}{2p-\beta}, \tag{2.20}$$

$$\left|\left(\frac{zf'(z)}{f(z)}\right)\frac{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1}{\frac{zf'(z)}{f(z)} - p}\right| < 1, \tag{2.21}$$

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1 + \delta\left(\frac{zf'(z)}{f(z)} - p\right)\right| < \frac{(p-\beta)[\delta(2p-\beta)+1]}{2p-\beta}. \tag{2.22}$$

then $f(z) \in \mathcal{S}^*(p, n, \beta)$.
The first four results (2.18) to (2.21) are also given recently by Prajapat ([4]).

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