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Fixed Points for Mappings Satisfying Occasionally Weakly Compatible Condition

H. K. Pathak[a] and Prachi Singh[1,b]

[1] **1.Abstract.** In this paper, we introduce the concept of pair-wise *occasionally weakly compatible* (*owc*) property for mappings in fuzzy metric spaces and prove some fixed point theorems by using a new implicit function of relation which involves a functional inequality relation between arguments.

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1 Introduction and Preliminaries

With the introduction of fuzzy set theory by L.Zadeh[1] and fuzzy metric space by Kamosil and Michalek[8] many authors fuzzify fixed points theorems proved in metric spaces. Initially Grabiac[3] fuzzify the well known Banach contraction principle to find fixed point in fuzzy metric space. After that many authors fuzzify the various contraction conditions satisfied by certain class of mappings such as commutative, weakly commutative, R-weakly commutative, compatible, weakly compatible etc. for a pair of mappings and proved many common fixed point theorems for such mappings.

In the same fashion we fuzzify the condition of occasionally weakly compatible for a pair of mappings in which the mappings are not necessarily commute at all points but they need to commute at least one coincidence point of mappings. The advantage of this is that we left the condition of completeness and subset-hood of any image space. No need of Cauchy sequence, which helps us to minimize the calculation and requirement of its limit.

To unify all contraction conditions V.Popa[10] introduced the notion of implicit functions. In this paper we extend the implicit functions and derive some related results as its examples.

Now we recall some notions and definitions in fuzzy metric spaces.

Definition 1.0.1 *[13]* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*
(a) $*$ *is associative and commutative,*
(b) $*$ *is continuous,*
(c) $a * 1 = a, \forall\, a \in [0,1]$,
(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

[1]Corresponding author

Definition 1.0.2 *[8]. A triplet $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty) \to [0, 1]$ satisfying, $\forall x, y \in X$, the following conditions:*
(1)$M(x, y, 0) = 0$,
(2)$M(x, y, t) = 1, \forall t > 0$ *iff* $x = y$,
(3)$M(x, y, t) = M(y, x, t)$,
(4)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s), t, s \in [0, 1)$,
(5)$M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous.*

Remark 1.0.3 *[3] It is easy to prove that $M(x, y, .)$ is non-decreasing for every $x, y \in X$.*

Remark 1.0.4 *If $(X, M, *)$ is a fuzzy metric space we can say that M is a fuzzy metric on X. Let (X, d) be a metric space. Let $a * b = ab$ for every $a, b \in [0, 1]$ and let $M_d : X \times X \times [0, \infty) \to [0, 1]$ be the function defined, for all $x, y \in X$ by $M_d(x, y, 0) = 0$ and for $t > 0$ by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

The triplet $(X, M_d, *)$ is a fuzzy metric space and M_d is called the fuzzy metric induced by d.

Definition 1.0.5 *Two self mappings A and B of fuzzy metric space $(X, M, *)$ are said to be*
*(1)***commutative** *if* $M(ASx, SAx, t) = 0 \;\; \forall \;\; t > 0.$
*(2)***weakly commutative** *[14] if*

$$M(ASx, SAx, t) \geq M(Ax, Sx, t) \;\; \forall \;\; t > 0$$

*(3)***compatible of type (A)***[5] if*
$\lim_{n\to\infty} M(ASx_n, SSx_n, t) = 1$ *and* $\lim_{n\to\infty} M(ASx_n, AAx_n, t) = 1 \; \forall \; t > 0$
whenever there exists a sequence $\{x_n\} \in X$ such that
$\lim_{n\to\infty} M(Sx_n, u, t) = \lim_{n\to\infty} M(Ax_n, u, t) \;\; \forall \;\; t > 0, \; u \in X$.
*(4)***compatible** *[6] if*
$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1 \;\; \forall \;\; t > 0$
whenever there exists a sequence $\{x_n\} \in X$ such that
$\lim_{n\to\infty} M(Sx_n, u, t) = \lim_{n\to\infty} M(Ax_n, u, t) \;\; \forall \;\; t > 0$ *for some* $u \in X$.
*(5)***weakly compatible** *[**?**] if they commute at their coincidence points*
i.e. $M(ABu, BAu, t) = 1$
whenever $M(Au, Bu, t) = 1 \;\; \forall \; t > 0$ *for some* $u \in X$.

2 Implicit relations

In this section we define two types of implicit relations and furnish various examples to verify them also with these examples define a variety of contraction conditions as corollaries. Let $\mathcal{F}$ be the set of all real-valued continuous functions F defined by
$F(t_1, t_2, t_3, t_4, t_5, t_6) : [0, 1]^6 \to [0, 1]$ satisfying the following conditions:
for $u, v \geq 0$,
(F_1) : $F(u, u, v, v, u, u) \leq 0$
then $u \geq r(v)$, $r : [0, 1] \to [0, 1]$ defined as
$r(t) > t$ and $r(0) = 0$, $r(1) = 1$.(**Example** $r(t) = \sqrt{t}$).

Example 2.0.6 *Define*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \phi(min\{t_2, t_3, t_4, t_5, t_6\}) - t_1$$

where $\phi(t) = r(t) > t \ \ \forall \ \ t \in (0,1)$ *and* $\phi(0) = r(0) = 0, \ \phi(1) = r(1) = 1$
then $(F_1) = F(u, u, v, v, u, u) = \phi(min\{u, v\}) - u \leq 0$
if and only if $u \geq \phi(min\{u, v\})$.
It is possible only when $min\{u, v\} = v$ *i.e.* $u \geq \phi(v) = r(v) > v$.
Also if $v = 1$ *then* $u \geq \phi(1) = 1$ *i.e.* $u = 1$.

Example 2.0.7 *Define*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = at_2 + bt_3 + ct_4 + max\{t_5, t_6\} - qt_1$$

$\forall \quad a, b, c \geq 0, \quad q > 0, \quad a + b + c + 1 > q$
then

$$\begin{aligned}(F_1): \quad F(u, u, v, v, u, u) &= au + bv + cv + max\{u, u\} - qu \\ &= (b + c)v - (q - a - 1)u \leq 0\end{aligned}$$

iff

$$u \geq \frac{b + c}{(q - a - 1)} v > v$$

i.e.

$$u \geq r(v) = \frac{b + c}{(q - a - 1)} v > v,$$

Example 2.0.8 *Define*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = at_2 + bt_3 + \frac{c \ max\{t_4, t_5\}}{t_6} - qt_1, \quad t_6 \neq 0$$

$\forall \quad a + b + c > q, \quad a, b, c \geq 0, \quad q > 0$ *then*

$$(F_1): F(u, u, v, v, u, u) = au + bv + \frac{c \ max\{v, u\}}{u} - qu,$$

If $v \geq u$ *then* $max\ \{v, u\} = v$ *then*

$$\begin{aligned}(q - a)u &\geq bv + \frac{cv}{u} \\ (q - a)u^2 &\geq buv + cv > bu^2 + cu \\ u &> \frac{c}{q - a - b} > 1\end{aligned}$$

which is absurd. Hence max $\{v, u\} = u$ *then*

$$(F_1) \ \ gives \ \ (q - a)u \geq (q - a)u^2 \geq bv + c$$

$$i.e. \ \ u \geq \frac{bv + c}{q - a} \ > \ v$$

$$u \geq r(v) = \tfrac{bv+c}{q-a} \ > v.$$

3 Main results

In this section we fuzzify the definition of occasionally weakly compatible(owc) which is introduced by M.A.Al-Thagafi and N.Shahzad[9] in metric space as $f(g(x)) = g(f(x))$ for some $x \in C(f, g) = \{x \in (X, d) | f(x) = g(x)\}$.

Definition 3.0.9 *Two self mappings A and B of fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible if they commute at one of their coincidence points i.e., there exists a point $u \in X$ such that*
$M(Au, Bu, t) = 1$ *then* $M(ABu, BAu, t) = 1 \quad \forall \ t > 0.$

Theorem 3.0.10 *Let $(X, M, *)$ be a fuzzy metric space and A, B, S, T be self-mappings of X satisfying the conditions:*
(4.2.1) mappings (A, S) and (B, T) are occasionally weakly compatible(owc).
(4.2.2) the inequality

$$F(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, t), M(Ax, Ty, t)) \leq 0$$

holds for all $x, y \in X$, $\forall \ t > 0$ and $F \in \mathcal{F}_6$ Then we have a unique common fixed point for the mappings A, B, S and T.

Proof Since (A, S) and (B, T) are owc, there exists points $u, v \in X$ such that $Au = su$ i.e. $M(Au, Su, t) = 1$ then $M(ASu, SAu, t) = 1$ and $Bv = Tv$ i.e. $M(Bv, Tv, t) = 1$ then $M(TBv, BTv, t) = 1$ for $t > 0$.

First we take that $Au \neq Bv$ i.e. $\forall t > 0$, $M(Au, Bv, t) \neq 1$ then by inequality (4.2.2)

$$F(M(Au, Bv, t), M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), M(Bv, Su, t), M(Au, Tv, t)) \leq 0$$

$$F(M(Au, Bv, t), M(Au, Bv, t), M(Au, Au, t), M(Bv, Bv, t), M(Bv, Au, t), M(Au, Bv, t)) \leq 0$$

$$F(M(Au, Bv, t), M(Au, Bv, t), 1, 1, M(Au, Bv, t), M(Au, Bv, t)) \leq 0$$

so F_1 gives $M(Au, Bv, t) \geq r(1) = 1$. Hence $Au = su = Bv = Tv$. Now suppose $A^2u \neq Au$ i.e. $\forall t > 0, M(A^2u, Au, t) \neq 1$ then by inequality (4.2.2)

$$F(M(A^2u, Bv, t), M(SAu, Tv, t), M(A^2u, SAu, t), M(Bv, Tv, t), M(Bv, SAu, t), M(A^2u, Tv, t)) \leq 0$$

$$F(M(A^2u, Au, t), M(ASu, Au, t), M(A^2u, ASu, t), M(Au, Au, t), M(Au, ASu, t), M(A^2u, Au, t)) \leq 0$$

$$F(M(A^2u, Au, t), M(A^2u, Au, t), M(A^2u, A^2u, t), M(Au, Au, t), M(Au, A^2u, t), M(A^2u, Au, t)) \leq 0$$

$$F(M(A^2u, Au, t), M(A^2u, Au, t), 1, 1, M(Au, A^2u, t), M(A^2u, Au, t)) \leq 0$$

so F_1 gives $M(Au, A^2u, t) \geq r(1) = 1$. Hence $A^2u = Au = su = Bv = Tv$. If we take $Au = z$ then $Az = z$. Similarly we can show that $Bz = Sz = Tz = z$.

Uniqueness Now for uniqueness of z let there is another point $w \in X$ such that $Aw = Bw = Tw = Sw = w$ and $w = z$ i.e. $\forall t > 0, M(z, w, t) \neq 1$ then by inequality (4.2.2)

$$F(M(Az, Bw, t), M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t), M(Bw, Sz, t), M(Az, Tw, t)) \leq 0$$

$$F(M(z, w, t), M(z, w, t), M(z, z, t), M(w, w, t), M(w, z, t), M(z, w, t)) \leq 0$$

$F(M(z,w,t), M(z,w,t), 1, 1, M(w,z,t), M(z,w,t)) \leq 0$
so F_1 gives $M(z,w,t) \geq r(1) = 1$. Hence $z = w$.
Hence the fixed point of all the functions is unique.

Remark 3.0.11 *Let $(X, M, *)$ be a fuzzy metric space and A, S be self-mappings of X such that (A,S) is owc and satisfying the condition*

$$F(M(Ax,Ay,t), M(Sx,Sy,t), M(Ax,Sx,t), M(Ay,Sy,t), M(Ay,Sx,t), M(Ax,Sy,t)) \leq 0$$

holds for all $x, y \in X$, $\forall$ $t > 0$ and $F \in \mathcal{F}_6$ then we have a unique common fixed point for the mappings A and S.

Corollary 3.0.12 *Let $(X, M, *)$ be a fuzzy metric space and A, B, S be self-mappings of X such that (A,S) and (B,S) are owc and satisfying the condition*

$$F(M(Ax,By,t), M(Sx,Sy,t), M(Ax,Sx,t), M(By,Sy,t), M(By,Sx,t), M(Ax,Sy,t)) \leq 0$$

holds for all $x, y \in X$, $\forall$ $t > 0$ and $F \in \mathcal{F}_6$ Then then we have a unique common fixed point for the mappings A, B and S.

Corollary 3.0.13 *Let $(X, M, *)$ be a fuzzy metric space and A, B, S, T be self-mappings of X such that (A,S) and (B,T) are owc and satisfying the inequality*

$$M(Ax,By,t) \geq \phi(min\{M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(By,Sx,t), M(Ax,Ty,t))\})$$

*where $\phi(t) = r(t) > t$ $\forall$ $t \in (0,1)$ and $\phi(0) = r(0) = 0$, $\phi(1) = r(1) = 1$
holds for all $x, y \in X$, $\forall$ $t > 0$ then A, B, S and T have common fixed point.*

Corollary 3.0.14 *Let $(X, M, *)$ be a fuzzy metric space and A, B, S, T be self-mappings of X such that ((A,S) and (B,T) are owc and satisfying the inequality*

$$q\ M(Ax,By,t) \geq a\ M(Sx,Ty,t) + b\ M(Ax,Sx,t), c\ M(By,Ty,t), + max\{M(By,Sx,t), M(Ax,Ty,t))\})$$

holds for all $x, y \in X$, $\forall$ $t > 0$ $\forall$ $a, b, c \geq 0$, $q > 0$, $a + b + c + 1 > q$ then A, B, S and T have common fixed point.

Corollary 3.0.15 *Let $(X, M, *)$ be a fuzzy metric space and A, B, S, T be self-mappings of X such that (A,S) and (B,T) are owc and satisfying the inequality*

$$q\ M(Ax,By,t) \geq a\ M(Sx,Ty,t) + \frac{b\ M(Ax,Sx,t) + c\ max\{M(Ax,Ty,t), M(By,Sx,t)\}}{M(By,Ty,t))}$$

holds for all $x, y \in X$, $\forall$ $t > 0$ $\forall$ $a, b, c \geq 0$, $q > 0$, $a + b + c > q$ then A, B, S and T have common fixed point.

Corollary 3.0.16 *Let $(X, M, *)$ be a fuzzy metric space and A, B, S, T be self-mappings of X such that (A,S) and (B,T) are owc and satisfying the inequality*

$$q\ M(Ax,By,t) \geq a\ M(Sx,Ty,t) + \frac{b\ M(By,Ty,t) + c\ max\{M(Ax,Ty,t), M(By,Sx,t)\}}{M(Ax,Sx,t))}$$

holds for all $x, y \in X$, $\forall \ t > 0 \ \forall \ a, b, c \geq 0$, $q > 0$, $a + b + c > q$ *then A, B, S and T have common fixed point.*

Corollary 3.0.17 *Let* $(X, M, *)$ *be a fuzzy metric space and* A, B, S, T *be self-mappings of* X *such that (A,S) and (B,T) are owc and satisfying the inequality*

$$q \ M(Ax, By, t) \geq b \ M(Ax, Sx, t) + c \ M(By, Ty, t) + \frac{a \ max\{M(Ax, Ty, t), M(By, Sx, t)\}}{M(Sx, Ty, t)}$$

holds for all $x, y \in X$, $\forall \ t > 0 \ \forall \ a, b, c \geq 0$, $q > 0$, $a + b + c > q$ *then A, B, S and T have common fixed point.*

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[a]School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur (C.G.) 492010, India
e-mail: hkpathak@sify.com
[1,b]Department of Mathematics,
Govt.V.Y.T.PG.Autonomous College
Durg (C.G.),491001,India
e-mail: prachibksingh@gmail.com