American J. of Mathematics and Sciences
Vol. 3, No -1 ,(January 2014)

ISSN No: 2250-3102

On a singular class of strongly indefinite Hamiltonian systems involving critical growth on a bounded set of $\mathbb{R}^2$

Brahim Khaldi, Nasreddine Megrez

Departement of Sciences, University of Bechar
PB 117, Bechar 08000, Algeria
email: khaldibra@yahoo.fr

Department of Mathematics, Prince Sultan University
P.O. Box 66833 , Riyadh, 11586, Saudi Arabia.
email: nmegrez@psu.edu.sa, nmegrez@gmail.com

December 30, 2013

Abstract

In this paper we study the existence of nontrivial solutions for the singular Hamiltonian elliptic system

$$\begin{cases} -\Delta u = \frac{g(v)}{|x|^a} & \text{in } \Omega \\ -\Delta v = \frac{f(u)}{|x|^a} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $a \in [0, 2)$ and the functions f and g have critical exponential grouth at $+\infty$. For the proof we use a variational argument (a linking theorem).

keywords: Hamiltonian system, Variational method, Trudinger-Moser inequality, strongly indefinite systems

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$ containing the origin. We consider the following Hamiltonian system of singular elliptic equations

$$\begin{cases} -\Delta u = \frac{g(v)}{|x|^a} & \text{in } \Omega \\ -\Delta v = \frac{f(u)}{|x|^a} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a \in [0,2)$, and the functions g and f satisfy the following:

H_1) $f,\ g : [0,+\infty[\to [0,+\infty[$ are continuous functions with $f = g = 0$ on $]-\infty,0]$, and $f(t) = \circ(t)$, $g(t) = \circ(t)$ near the origin.

H_2) There exist constants $\theta > 2$ and $t_0 > 0$ such that

$$0 < \theta F(t) \leq tf(t) \ \text{ and } \ 0 < \theta G(t) \leq tg(t) \quad \forall t \geq t_0,$$

where $F(t) = \int_0^t f(s)\,ds$ and $G(t) = \int_0^t g(s)\,ds$.

H_3) There exist $M > 0$ and $R > 0$ such that for all $t \geq R$

$$0 < F(t) \leq Mf(t) \ \text{ and } \ 0 < G(t) \leq Mg(t).$$

H_4) There exists $\beta_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{tf(t)}{e^{\beta_0 t^2}} > \frac{(2-a)^2}{\beta_0 d^{2-a}}, \text{ and } \lim_{t\to+\infty} \frac{tg(t)}{e^{\beta_0 t^2}} > \frac{(2-a)^2}{\beta_0 d^{2-a}},$$

where d is the radius of the largest open ball centred at origin and contained in Ω.

H_5) $\forall\epsilon > 0$ there exists positive constant C_ϵ such that

$$f(t) \leq C_\epsilon e^{(\beta_0+\epsilon)t^2},\ g(t) \leq C_\epsilon e^{(\beta_0+\epsilon)t^2},\ \forall t \geq 0.$$

Hypothesis H_4) implies that f and g have critical growth at $+\infty$.

We say that a function f has critical growth at $+\infty$ if there exists $\beta_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{f(t)}{e^{\beta t^2}} = \begin{cases} 0, & \text{for all } \beta > \beta_0 \\ +\infty, & \text{for all } \beta < \beta_0. \end{cases}$$

This notion of criticality is motivated by Trudinger-Moser inequality (see [13],[18]) which says that if $u \in H_0^1(\Omega)$ then $e^{\beta u^2} \in L^1(\Omega)$. Moreover, there exists a constant $C > 0$ such that

$$\sup_{\|u\|\leq 1} \int_\Omega e^{\beta u^2} dx \leq C\,|\Omega|\,, \quad \text{if } \beta \leq 4\pi.$$

Problems of the type (1.1) with nonlinearity having polynomial growth have been studied in [4], [8] and [10] in the case $a \neq 0$, and by De Figuerido and Felmer [5], Dai and Gu [3], and Hulshof et al. [11] in the case $a = 0$.

System (1.1) involving critical or subcritical exponential growth and without weights ($a = 0$) have been investigated in [7], [9] and [15]. In [19], a Schrodinger version of system (1.1) has been studied on the whole space $\mathbb{R}^2$, where a compact Sobolev embedding was recovered by the presence of a potential bounded away from 0 and whose the inverse is bounded in $L^1(\mathbb{R}^2)$.

Our work in this paper is closely related to the work in [17] where the authors studied the Gradient system

$$i \in \mathbb{N},\ -\Delta u_i = \frac{\partial \tilde{F}}{\partial u_i}(x, u_1, \ldots, u_m) + h_i(x) \text{ in } \Omega,$$

which is reduced to

$$\begin{cases} -\Delta u = \dfrac{\partial \tilde{F}}{\partial u} & \text{in } \Omega \\ -\Delta v = \dfrac{\partial \tilde{F}}{\partial v} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

if $m = 2$ and $h_i \equiv 0$.

Note that our system (1.1) is considered as a Hamiltonian (not Gradient) system since if we write

$$H(u, v) := \frac{F(u)}{|x|^a} + \frac{G(v)}{|x|^a},$$

then, system (1.1) takes the form

$$\begin{cases} -\Delta u = \dfrac{\partial H}{\partial v} & \text{in } \Omega \\ -\Delta v = \dfrac{\partial H}{\partial u} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

showing the structural difference between our problem and the one studied in [17]. Our work is then seen as an extension of [17] to the case of Hamiltonian systems involving critical growth. It can also be considered as an extension of the results in [7] for the critical case from $a = 0$ to $a \in [0, 2)$ where the limitation on a is due to Lemma 2.1.

Unlike [17], the strongly indefinte character of the functional associated to (1.1) does not allow us to use classical Mountain Pass results and we shall use linking methods instead, as in [7]. The presence of the singular term $|x|^{-a}$ prevents us from using the classical Trudinger-Moser inequality, and an adapted version of the Trudinger-Moser inequality with singular weight due to Adimurthi-Sandeep [2] (see Lemma 2.1 in the next section) will be the key tool to handle the singular nonlinearity.

We are interested in finding nontrivial solutions of (1.1) in the space $E := H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the norm

$$\|(u,v)\|_E := \left(\|u\|^2 + \|v\|^2 \right)^{\frac{1}{2}},$$

where $\|u\| = \left(\int\limits_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}$ is the norm of the Sobolev space $H_0^1(\Omega)$.

Note that $f,\ g$ have maximal growth, which allows us to treat the problem (1.1) variationaly in E. It is then natural to find the solutions of our problem by looking for critical points of the corresponding functional

$$I(u,v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega \frac{F(u)}{|x|^a} dx - \int_\Omega \frac{G(u)}{|x|^a} dx,$$

in the space $E := H_0^1(\Omega) \times H_0^1(\Omega)$. Under our assumptions, this functional is well defined and $C^1(E, \mathbb{R})$. Also, for all $(\varphi, \psi) \in E$, we have

$$I'(u,v)(\varphi,\psi) = \int_\Omega \nabla u \nabla \psi dx + \int_\Omega \nabla v \nabla \varphi dx - \int_\Omega \frac{f(u)\varphi}{|x|^a} dx - \int_\Omega \frac{g(v)\psi}{|x|^a} dx.$$

The main result in this paper is the following theorem

Theorem 1.1. *If (H_1), (H_2), (H_3), (H_4), and $H_5)$ are satisfied, then problem (1.1) has a nontrivial weak solution $(u,v) \in E$.*

2 preliminaries

In this paper, we shall use the following version of Trudinger-Moser inequality with a singular weight due to Adimurthi-Sandeep [2].

Lemma 2.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *containing* 0 *and* $u \in H_0^1(\Omega)$. *Then, for every* $\alpha > 0$ *and* $a \in [0, 2)$

$$\int_{\Omega} \frac{e^{\alpha u^2}}{|x|^a} dx < \infty.$$

Moreover,

$$\sup_{\|u\| \leq 1} \int_{\Omega} \frac{e^{\alpha u^2}}{|x|^a} dx < \infty \tag{2.1}$$

if and only if $\frac{\alpha}{4\pi} + \frac{a}{2} \leq 1$.

To show that the Palais-Smale sequence is bounded in E, we will use the following inequality proved in [7]:

Lemma 2.2. *The following inequality holds*

$$st \leq \begin{cases} \left(e^{t^2} - 1\right) + s\left(\log^+ s\right)^{\frac{1}{2}}, & \text{for } t \geq 0 \text{ and } s \geq e^{\frac{1}{4}} \\ \left(e^{t^2} - 1\right) + \frac{1}{2}s^2, & \text{for } t \geq 0 \text{ and } s \leq e^{\frac{1}{4}} \end{cases} \tag{2.2}$$

Lemma 2.3. *Let* $u \in H_0^1(\Omega)$ *and* $a \in [0, 2)$. *Then there exist* $C > 0$ *such that*

$$\int_{\Omega} \frac{|u|^2}{|x|^a} dx \leq C \|u\|^2 \tag{2.3}$$

Proof. Using Hölder's inequality, we have

$$\int_{\Omega} \frac{|u|^2}{|x|^a} dx \leq \left(\int_{\Omega} |x|^{\frac{-ar}{r-2}} dx\right)^{\frac{r-2}{r}} \left(\int_{\Omega} |u|^r dx.\right)^{\frac{2}{r}}.$$

We can choose r such that $r > \frac{4}{2-a}$. Therefore,

$$\int_{\Omega} \frac{|u|^2}{|x|^a} dx \leq C \|u\|_r^2.$$

Finally, by the continuous embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$, we conclude that

$$\int_\Omega \frac{|u|^2}{|x|^a} dx \leq C \|u\|^2.$$

□

We will also use the following convergence result (Lemma 4.2 in [17]):

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, for any sequence (u_n) in $L^1(\Omega)$ such that*

$$u_n \to u \ \textit{in} \ L^1(\Omega), \quad \frac{f(x,u_n)}{|x|^a} \in L^1(\Omega), \quad \textit{and} \quad \int_\Omega \frac{|f(x,u_n)\,u_n|}{|x|^a} dx \leq C,$$

up to a subsequence we have

$$\frac{f(x,u_n)}{|x|^a} \to \frac{f(x,u)}{|x|^a} \quad \textit{in} \quad L^1(\Omega)$$

Lemma 2.5. *Let (u_n, v_n) be a Palais-Smale sequence for the fonctional I such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. Then (u_n, v_n) has a subsequence, still denoted by (u_n, v_n) such that*

$$\frac{F(u_n)}{|x|^a} \to \frac{F(u)}{|x|^a} \ \textit{in} \ L^1(\Omega) \quad \textit{and} \quad \frac{G(v_n)}{|x|^a} \to \frac{G(v)}{|x|^a} \ \textit{in} \ L^1(\Omega).$$

Proof. From (H_3), we can conclude that

$$|F(u_n)| \leq M_1 + M\,|f(u_n)| \quad \text{and} \quad |G(v_n)| \leq M_2 + M\,|g(v_n)| \tag{2.4}$$

where $M_1 = \sup\limits_{[-R,R]} |F(u_n)|$, and $M_2 = \sup\limits_{[-R,R]} |G(v_n)|$.

On the other hand, from Lemma 2.4, we have

$$\frac{f(u_n)}{|x|^a} \to \frac{f(u)}{|x|^a} \text{ in } L^1(\Omega), \text{ and } \frac{g(v_n)}{|x|^a} \to \frac{g(v)}{|x|^a} \text{ in } L^1(\Omega),$$

which implies that there exist h_1, $h_2 \in L^1(\Omega)$ such that

$$\frac{|f(u_n)|}{|x|^a} \leq h_1 \ \text{ and } \ \frac{|g(v_n)|}{|x|^a} \leq h_2 \text{ almost everywhere in } \Omega \tag{2.5}$$

Then, by (2.4), (2.5) and Lebesgue dominated convergence Theorem, we get

$$\frac{F(u_n)}{|x|^a} \to \frac{F(u)}{|x|^a} \text{ in } L^1(\Omega), \text{ and } \frac{G(v_n)}{|x|^a} \to \frac{G(v)}{|x|^a} \text{ in } L^1(\Omega).$$

□

Remark 2.6. C is a generic positive constant.

3 Linking structure and Plais-Smale sequences

Since the energy functional I has strong indefinite quadratic part, we cannot use classical min-max methods. Instead, we use linking theory to give a Palais-Smale sequence by the minimax principle used in [14]:

Definition 3.1. Let S be a closed subset of a Banach space X, and Q a sub-manifold of X, with relative boundary ∂Q.
We say that S and ∂Q link if:

1. $S \cap \partial Q = \emptyset$.
2. $\forall h \in C^0(X, X)$ such that $h_{|\partial Q} = id$, there holds $h(Q) \cap S \neq \emptyset$.

Theorem 3.2. *Let $J : X \longrightarrow \mathbb{R}$ be a C^1 functional. Consider a closed subset $S \subset X$, and a sub-manifold $Q \subset X$, with relative boundary ∂Q. Suppose:*

1. *S and ∂Q link.*
2. *$\exists \delta > 0$ such that*

$$J(z) \geq \delta \ \forall z \in S,$$
$$J(z) \leq 0 \ \forall z \in \partial Q.$$

Let

$$\Gamma := \{h \in C^0(X, X) \mid h_{|\partial Q} = id\},$$

and

$$c := \inf_{h \in \Gamma} \sup_{z \in Q} J(h(z)) \geq \delta.$$

Then, there exists a sequence $(z_k)_{k \in \mathbb{N}} \subset X$, such that

$$\begin{cases} J(z_k) \xrightarrow[k \to \infty]{} c, \\ J'(z_k) \xrightarrow[k \to \infty]{} 0. \end{cases} \tag{3.1}$$

To verify that the functional I has a linking structure (i.e. satisfies (2) in the previous Theorem), we use the following notations:

$$E^{+}=\left\{(u,u)\mid u\in H_0^1(\Omega)\right\} \text{ and } E^{-}=\left\{(u,-u)\mid u\in H_0^1(\Omega)\right\},$$

$$S:=\{(u,u)\in E^{+}\mid \|(u,u)\|=\rho\}=\partial B_\rho\cap E^{+},$$

and

$$Q:=\left\{r\,(e,e)+\omega:\omega\in E^{-},\ \ \|\omega\|\le R_0 \text{ and } 0\le r\le R_1\right\}\subset\mathbb{R}(e,e)\oplus E^{-},$$

where $e\in H_0^1(\Omega)$ is a fixed nonnegative function with $\|e\|=1$.

Lemma 3.3. *There exist $\rho>0$ and $\sigma>0$ such that*

$$I(z)\ge\sigma,\ \textit{for all } z\in S.$$

Proof. From (H_1), for a given $\varepsilon>0$ there exists t_0 such that

$$f(t)\le 2\varepsilon t \quad\text{and}\quad g(t)\le 2\varepsilon t,\quad \text{for all } t\le t_0 \tag{3.2}$$

In the other hand, it follows from H_5) that for a given $q>2$, there exists a constant $C>0$ and β such that

$$F(t)\le Ct^q e^{\beta t^2},\ \text{and } G(t)\le Ct^q e^{\beta t^2},\quad \text{for all } t\ge t_0 \tag{3.3}$$

From (3.2) and (3.3), we get

$$F(t)\le\varepsilon t^2+Ct^q e^{\beta t^2}\quad\text{and}\quad G(t)\le\varepsilon t^2+Ct^q e^{\beta t^2},\quad \text{for all } t\ge 0 \tag{3.4}$$

Now, for $z\in S$, we have

$$I(z)=\int_\Omega|\nabla u|^2\,dx-\int_\Omega\frac{F(u)}{|x|^a}dx-\int_\Omega\frac{G(u)}{|x|^a}dx.$$

Using (3.4), (2.3), and Hölder inequality, we get

$$\begin{aligned}
I(z)&\ge\|u\|^2-2\varepsilon\int_\Omega\frac{u^2}{|x|^a}dx-2C\int_\Omega\frac{u^q e^{\beta u^2}}{|x|^a}dx\\
&\ge(1-C\varepsilon)\|u\|^2-2C\left(\int_\Omega u^{qs'}dx\right)^{\frac{1}{s'}}\left(\int_\Omega\frac{e^{s\beta u^2}}{|x|^{as}}dx\right)^{\frac{1}{s}}\\
&\ge(1-C\varepsilon)\|u\|^2-2C\,\|u\|_{qs'}^q\left(\int_\Omega\frac{e^{s\|u\|^2\beta\left(\frac{u}{\|u\|}\right)^2}}{|x|^{as}}dx\right)^{\frac{1}{s}},
\end{aligned}$$

where $\frac{1}{s'} + \frac{1}{s} = 1$ with s sufficiently close to 1 such that $as < 2$ and $qs' > 1$.

Now, for $\|u\| \leq \delta$, with $\delta > 0$ such that $\frac{\beta s \delta^2}{4\pi} + \frac{as}{2} \leq 1$, by Trudinger-Moser inequality (2.1) and Sobolev imbedding Theorem we obtain

$$I(z) \geq (1 - C\varepsilon) \|u\|^2 - 2C \|u\|^q .$$

Then, for ε small enough we can find $\rho, \sigma > 0$ such that $I(z) \geq \sigma > 0$ for $\|u\| = \rho$ sufficiently small. □

Lemma 3.4. *There exist R_0, $R_1 > 0$ such that $I(z) \leq 0$ for all $z \in \partial Q$, where ∂Q denotes the boundary of Q in $\mathbb{R}(e,e) \oplus E^-$.*

Proof. For $z \in \partial Q$, we have three cases:

Case 1: $z \in \partial Q \cap E^-$. We have $z = (u, -u)$ and hence

$$I(z) = -\int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{F(u)}{|x|^a} dx - \int_\Omega \frac{G(-u)}{|x|^a} dx \leq -\|u\|^2 \leq 0.$$

Case 2: $z = R_1(e,e) + (u, -u) \in \partial Q$ with $\|(u, -u)\| \leq R_0$. Then

$$I(z) = R_1^2 - \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{F(R_1 e + u)}{|x|^a} dx - \int_\Omega \frac{G(R_1 e - u)}{|x|^a} dx \qquad (3.5)$$

By the assumption (H_2), there exists $C > 0$ such that

$$F(t) \geq C\left(t^\theta - 1\right), \text{ and } G(t) \geq C\left(t^\theta - 1\right).$$

We then obtain from (3.5) that

$$I(z) \leq R_1^2 - C \int_\Omega \frac{(R_1 e + u)^\theta + (R_1 e - u)^\theta}{|x|^a} dx + C.$$

Now, using the convexity of the function $\phi(t) = t^\theta$, it follows that

$$I(z) \leq R_1^2 - 2C R_1^\theta \int_\Omega \frac{e^\theta}{|x|^a} dx + C.$$

Then, for R_1 sufficiently large, we get $I(z) \leq 0$.

Case 3: $z = r(e,e) + (u, -u) \in \partial Q$ with $\|(u, -u)\| = R_0$ and $0 \leq r \leq R_1$. Then,

$$\begin{aligned} I(z) &= r^2 - \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{F(re + u)}{|x|^a} dx - \int_\Omega \frac{G(re - u)}{|x|^a} dx \\ &\leq R_1^2 - \frac{1}{2} R_0^2. \end{aligned}$$

Thus, $I(z) \leq 0$ if $R_0 \geq \sqrt{2} R_1$. □

To prove that a Palais-Smale sequence converges to a weak solution of problem (1.1), we need to establish the following Lemma:

Lemma 3.5. *Let* $(u_n, v_n) \in E$ *such that* $I(u_n, v_n) \to c$ *and* $I'(u_n, v_n) \to 0$. *Then,*

$$\|u_n\| \le C, \qquad \|v_n\| \le C \tag{3.6}$$

$$\int_\Omega \frac{f(u_n)\,u_n}{|x|^a}dx \le C, \quad \int_\Omega \frac{g(v_n)\,v_n}{|x|^a}dx \le C \tag{3.7}$$

$$\int_\Omega \frac{F(u_n)}{|x|^a}dx \le C, \qquad \int_\Omega \frac{G(v_n)}{|x|^a}dx \le C \tag{3.8}$$

Proof. Let $(u_n, v_n) \in E$ be a sequence such that $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$, that is,

$$\int_\Omega \nabla u_n \nabla v_n dx - \int_\Omega \frac{F(u_n)}{|x|^a}dx - \int_\Omega \frac{G(v_n)}{|x|^a}dx = c + \delta_n \tag{3.9}$$

and for any $(\varphi, \psi) \in E$,

$$\left|\int_\Omega \nabla u_n \psi dx + \int_\Omega \nabla\varphi\nabla v_n dx - \int_\Omega \frac{f(u_n)\,\varphi}{|x|^a}dx - \int_\Omega \frac{g(v_n)\,\psi}{|x|^a}dx\right| \le \varepsilon_n \|(\varphi,\psi)\|. \tag{3.10}$$

Choosing $(\varphi, \psi) = (u_n, v_n)$ in (3.10) and using (H_2), we have

$$\begin{aligned}\int_\Omega \frac{f(u_n)\,u_n}{|x|^a}dx + \int_\Omega \frac{g(v_n)\,v_n}{|x|^a}dx &\le 2\left|\int_\Omega \nabla u_n \nabla v_n dx\right| + \varepsilon_n \|(u_n, v_n)\| \\ &\le 2c + 2\int_\Omega \frac{F(u_n)}{|x|^a}dx + 2\int_\Omega \frac{G(v_n)}{|x|^a}dx + 2\delta_n + \\ &+ \varepsilon_n \|(u_n, v_n)\| \\ &\le 2c + \frac{2}{\theta}\int_\Omega \frac{f(u_n)\,u_n}{|x|^a}dx + \frac{2}{\theta}\int_\Omega \frac{g(v_n)\,v_n}{|x|^a}dx + 2\delta_n \\ &+ \varepsilon_n \|(u_n, v_n)\|.\end{aligned}$$

Thus,

$$\int_\Omega \frac{f(u_n)\,u_n}{|x|^a}dx + \int_\Omega \frac{g(v_n)\,v_n}{|x|^a}dx \le C(1 + 2\delta_n + \varepsilon_n \|(u_n, v_n)\|) \tag{3.11}$$

Now, taking $(\varphi, \psi) = (v_n, 0)$ and $(\varphi, \psi) = (0, u_n)$ in (3.10), we get

$$\|v_n\|^2 - \varepsilon_n \|v_n\| \leq \int_\Omega \frac{f(u_n)\, v_n}{|x|^a} dx \tag{3.12}$$

and

$$\|u_n\|^2 - \varepsilon_n \|u_n\| \leq \int_\Omega \frac{g(v_n)\, u_n}{|x|^a} dx \tag{3.13}$$

Setting $V_n = \dfrac{v_n}{\|v_n\|}$ and $U_n = \dfrac{u_n}{\|u_n\|}$, we obtain

$$\|v_n\| \leq \int_\Omega \frac{f(u_n)}{|x|^a} V_n dx + \varepsilon_n, \text{ and } \ \|u_n\| \leq \int_\Omega \frac{g(v_n)}{|x|^a} U_n dx + \varepsilon_n \tag{3.14}$$

Using inequality (2.2) with $t = V_n$ and $s = f(u_n)$, in the first estimate in (3.14), we obtain

$$\int_\Omega \frac{f(u_n)}{|x|^a} V_n dx \leq C \int_\Omega \frac{e^{V_n^2}}{|x|^a} dx + \int_{\left\{x \in \Omega:\ f(u_n) \geq e^{\frac{1}{4}}\right\}} \frac{f(u_n)}{|x|^a} \left[\log\left(f(u_n)\right)\right]^{\frac{1}{2}} dx +$$
$$+ \frac{1}{2} \int_{\left\{x \in \Omega:\ f(u_n) \leq e^{\frac{1}{4}}\right\}} \frac{\left[f(u_n)\right]^2}{|x|^a} dx.$$

Using Trudinger-Moser inequality and the fact $a < 2$, we get

$$\int_\Omega \frac{f(u_n)}{|x|^a} V_n dx \leq C \left(1 + \beta^{\frac{1}{2}} \int_\Omega \frac{f(u_n)\, u_n}{|x|^a} dx\right).$$

This estimate together with the first inequality in (3.14) imply that

$$\|v_n\| \leq C \left(1 + \int_\Omega \frac{f(u_n)\, u_n}{|x|^a} dx + \varepsilon_n\right) \tag{3.15}$$

Similary, we get from the second estimate in (3.14)

$$\|u_n\| \leq C \left(1 + \int_\Omega \frac{g(v_n)\, v_n}{|x|^a} dx + \varepsilon_n\right) \tag{3.16}$$

Adding the estimates (3.15) and (3.16) and using (3.11), we obtain

$$\|(u_n, v_n)\| \leq C \left(1 + \delta_n + \varepsilon_n \|(u_n, v_n)\| + \varepsilon_n\right)$$

Then, $\|(u_n, v_n)\| \leq C$. From this estimate, inequality (3.11) and (H_2), we obtain the estimates (3.7) and (3.8), which completes the proof. □

4 Proof of the main result

4.1 Finite-dimensional approximation

Since $\dim E^{\pm} = \infty$, the functional I is strongly indefinite and all of its critical points have infinite Morse index. Thus, the standard linking theorems can not be applied. We therefore approximate problem (1.1) by a sequence of finite dimensional spaces (Galerkin approximation).

Denote by $(\phi_i)_{i\in\mathbb{N}}$ an orthonormal set of eigenfunctions corresponding to the eigenvalues (λ_i), $i \in \mathbb{N}$, of $(-\Delta, H_0^1(\Omega))$ and set

$$\begin{aligned} E_n^+ &= span\{(\phi_i, \phi_i) \mid i = 1, ..., n\} \\ E_n^- &= span\{(\phi_i, -\phi_i) \mid i = 1, ..., n\} \\ E_n &= E_n^+ \oplus E_n^- \end{aligned}$$

Set now $Q_n := Q \cap E_n \subset \mathbb{R}(e,e) \oplus E_n^-$, where Q as in previous section, and define the class of mappings

$$\Gamma_n = \{\gamma \in C(Q_n, \mathbb{R}(e,e) \oplus E_n) : \ \gamma(z) = z \ \text{ on } \ \partial Q_n\}$$

and set

$$c_{n,e} = \inf_{\gamma\in\Gamma_n} \max_{z\in Q_n} I(\gamma(z)) \tag{4.1}$$

Using an intersection Theorem (Proposition 5.9 in [14]), we have

$$\gamma(Q_n) \cap S \neq \emptyset, \quad \forall\, \gamma \in \Gamma_n,$$

which, in combination with Lemma 3.3, imply that

$$c_{n,e} \geq \sigma > 0.$$

On the other hand, since the identity mapping $Id : Q_n \to \mathbb{R}(e,e) \oplus E_n$ belongs to Γ_n, it is easy to prove that $c_{n,e} \leq R_1^2$. Then, we have

$$0 < \sigma \leq c_{n,e} \leq R_1^2$$

Now, by Lemma 3.3 and Lemma 3.4, we see that the linking geometry holds for the functional $I_n = I|_{E_n}$. Therefore, applying the linking Theorem for I_n (see theorem 5.3 in [14]), we get the following result:

For each $n \in \mathbb{N}$ the functional I_n has a critical point $z_n = (u_n, v_n) \in E_n$ at level c_n such that

$$I_n(z_n) = c_{n,e} \in \left[\sigma, R_1^2\right] \tag{4.2}$$

and

$$I_n'(z_n) = 0.$$

Furthermore, $\|z_n\| \leq C$ where C does not depend in n.

4.2 On the mini-max level

In order to get a more precise information about the minimax level, we consider for $k \in \mathbb{N}$, the sequence

$$\tilde{\psi}_k(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} (\log k)^{1/2} & \text{for } 0 \leq |x| \leq \frac{1}{k} \\ \frac{\log \frac{1}{|x|}}{(\log k)^{1/2}} & \text{for } \frac{1}{k} \leq |x| \leq 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

and by setting $e_k(x) = \tilde{\psi}_k(\frac{x}{d})$, we define the sets

$$Q_{n,k} = \left\{ r\left(e_k, e_k\right) + \omega : \omega \in E_n^-,\ \ \|\omega\| \leq R_0 \text{ and } 0 \leq r \leq R_1 \right\},$$

Lemma 4.1. *There exists $k \in \mathbb{N}$ such that*

$$\sup_{\mathbb{R}_+(e_k, e_k) \oplus E^-} I < \frac{2\pi\,(2-a)}{\beta_0}.$$

Proof. Suppose by contradiction that for all $k \in \mathbb{N}$, we have

$$\sup_{\mathbb{R}_+(e_k, e_k) \oplus E^-} I \geq \frac{2\pi\,(2-a)}{\beta_0}.$$

This means that there exists $z_{n,k} = \tau_{n,k}\,(e_k, e_k) + (u_{n,k}, -u_{n,k}) \in Q_{n,k}$ such that

$$I\left(z_{n,k}\right) \geq \frac{2\pi\,(2-a)}{\beta_0} - \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Let $h(t) := I(tz_{n,k})$. We see that $h(0) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$. Then, there exixts a maximum point $t_0 z_{n,k}$ with $I(t_0 z_{n,k}) \geq \frac{2\pi(2-a)}{\beta_0} - \varepsilon_n$. We may assume that $z_{n,k}$ is this point, and then we get

$$\tau_{n,k}^2 - \int_\Omega |\nabla u_{n,k}|^2 \, dx - \int_\Omega \frac{F(\tau_{n,k}e_k + u_{n,k})}{|x|^a} dx - \int_\Omega \frac{G(\tau_{n,k}e_k - u_{n,k})}{|x|^a} dx \geq \frac{2\pi(2-a)}{\beta_0} - \varepsilon_n \tag{4.2}$$

and

$$\tau_{n,k}^2 - \int_\Omega |\nabla u_{n,k}|^2 \, dx = \int_\Omega \frac{f(\tau_{n,k}e_k + u_{n,k})(\tau_{n,k}e_k + u_{n,k}) - g(\tau_{n,k}e_k - u_{n,k})(\tau_{n,k}e_k - u_{n,k})}{|x|^a} dx \tag{4.3}$$

Now, put $\tau_{n,k}^2 = s_n + \frac{2\pi(2-a)}{\beta_0}$. So, from (4.2) we get $s_n + \frac{2\pi(2-a)}{\beta_0} \geq \frac{2\pi(2-a)}{\beta_0} - \varepsilon_n$.

By assumption (H_4), there exists $\bar{t} > 0$ and

$$\eta_0 > \frac{(2-a)^2}{\beta_0 d^{2-a}} \tag{4.5}$$

such that

$$tf(t) \geq (\eta_0 - \varepsilon) e^{\beta_0 t^2}, \text{ and } tg(t) \geq (\eta_0 - \varepsilon) e^{\beta_0 t^2}, \tag{4.6}$$

for all $t \geq \bar{t}$ and ε is arbitrarly small.

Next, choosing k sufficiently large such that $\tau_{n,k}\sqrt{\frac{(\log k)}{2\pi}} \geq \bar{t}$, we get

$$\max\{\tau_{n,k}e_k + u_{n,k}, \tau_{n,k}e_k - u_{n,k}\} \geq \bar{t} \text{ for all } x \in B_{\frac{d}{k}}(0).$$

Now, using (4.3) and (4.6), we obtain

$$\begin{aligned} s_n + \frac{2\pi(2-a)}{\beta_0} &\geq (\eta_0 - \varepsilon) \int_{B_{\frac{d}{k}}(0)} \frac{e^{\beta_0 \tau_{n,k}^2 \frac{(\log k)}{2\pi}}}{|x|^a} dx \\ &\geq (\eta_0 - \varepsilon) 2\pi e^{\beta_0\left(s_n + \frac{2\pi(2-a)}{\beta_0}\right)\frac{(\log k)}{2\pi}} \int_0^{\frac{d}{k}} \xi^{1-a} d\xi \\ &\geq (\eta_0 - \varepsilon) 2\pi e^{\beta_0 s_n \frac{(\log k)}{2\pi}} e^{(2-a)(\log k)} \left(\frac{d}{k}\right)^{2-a} \\ &\geq (\eta_0 - \varepsilon) \frac{2\pi d^{2-a} e^{\beta_0 s_n \frac{(\log k)}{2\pi}}}{2-a}. \end{aligned}$$

This and (4.2) imply that $\lim_{n\to+\infty} s_n = 0$. So, we see that $(\eta_0 - \varepsilon) \leq \frac{2(2-a)^2}{\beta_0 d^{2-a}}$, which contradicts (4.5). □

4.3 Proof of Theorem 1

Lemma 4.1 implies that there is $\delta > 0$ suh that

$$c_n := c_{n,e} \leq \frac{2\pi(2-a)}{\beta_0} - \delta$$

where $c_{n,e}$ is defined by (4.1).

Next, using (4.2) and Lemma 3.5, we have $z_n = (u_n, v_n) \in E_n$ bounded in E such that

$$I_n(z_n) = c_n \in \left[\sigma, \frac{2\pi(2-a)}{\beta_0} - \delta\right], \tag{4.7}$$

$$I_n'(z_n) = 0, \tag{4.8}$$

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in} \quad E,$$

$$u_n \to u \quad \text{and} \quad v_n \to v \quad \text{in} \quad L^q(\Omega), \ \forall\, q \geq 1,$$

$$u_n(x) \to u(x) \quad \text{and} \quad v_n(x) \to v \quad \text{a. e. in } \Omega$$

By Lemma 3.5, we have

$$\int_\Omega \frac{f(u_n)\,u_n}{|x|^a} dx \leq C, \quad \int_\Omega \frac{g(v_n)\,v_n}{|x|^a} dx \leq C \tag{4.4}$$

$$\int_\Omega \frac{F(u_n)}{|x|^a} dx \leq C, \qquad \int_\Omega \frac{G(v_n)}{|x|^a} dx \leq C \tag{4.5}$$

Taking as test functions $(0, \psi)$ and $(\varphi, 0)$ in (4.8), where φ and ψ are arbitrary functions in $F_n := span\{\phi_i : i = 1, ..., n\}$, we get

$$\int_\Omega \nabla u_n \nabla \psi dx = \int_\Omega \frac{g(v_n)\,\psi}{|x|^a} dx \quad \forall \psi \in F_n \tag{4.6}$$

$$\int_\Omega \nabla v_n \nabla \varphi dx = \int_\Omega \frac{f(u_n)\,\varphi}{|x|^a} dx \quad \forall \varphi \in F_n \tag{4.7}$$

Consequently, by Lemma 3.5 and 2.4, $\frac{f(u_n)}{|x|^a} \to \frac{f(u)}{|x|^a}$ and $\frac{g(v_n)}{|x|^a} \to \frac{g(v)}{|x|^a}$ in $L^1(\Omega)$. Passing to the limit in (4.6) and (4.7) and using the fact that $\bigcup_{n\in\mathbb{N}} F_n$ is dense in $H_0^1(\Omega)$, we see that

$$\int_\Omega \nabla u \nabla \psi dx = \int_\Omega \frac{g(v)\,\psi}{|x|^a} dx \quad \forall \psi \in H_0^1(\Omega) \tag{4.8}$$

$$\int_\Omega \nabla v \nabla \varphi dx = \int_\Omega \frac{f(u)\,\varphi}{|x|^a} dx \quad \forall \varphi \in H_0^1(\Omega) \tag{4.9}$$

Thus, we conclude that (u, v) is a weak solution of (1.1).

Finally, it only remains to prove that $(u, v) \in E$ is nontrivial. Assume by contradiction that $u = 0$, which implies that also $v = 0$. Now, if $\|u_n\| \to 0$, then we get directly (4.15) below, and then a contraduction. Thus, asume that $\|u_n\| \geq b > 0, \ \forall n$ and consider

$$\|u_n\|^2 = \int_\Omega \frac{g(v_n)\,u_n}{|x|^a} dx \tag{4.10}$$

Setting $\overline{u}_n = \left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \frac{u_n}{\|u_n\|}$, and using inequality (2.2) with $s = \frac{g(v_n)}{\sqrt{\beta_0}}$ and $t = \sqrt{\beta_0}\,\overline{u}_n$, we have

$$\begin{aligned}\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \|u_n\| &= \int_\Omega \frac{g(v_n)\,\overline{u}_n}{|x|^a} dx \\ &\leq \int_\Omega \frac{e^{\beta_0 \overline{u}_n^2} - 1}{|x|^a} dx + \int_{\left\{x\in\Omega: \frac{g(v_n(x))}{\sqrt{\beta_0}} \leq e^{\frac{1}{4}}\right\}} \frac{(g(v_n))^2}{\beta_0 |x|^a} dx \\ &+ \int_{\left\{x\in\Omega: \frac{g(v_n(x))}{\sqrt{\beta_0}} \geq e^{\frac{1}{4}}\right\}} \frac{g(v_n)}{\sqrt{\beta_0}\,|x|^a} \left(\log\left(\frac{g(v_n)}{\sqrt{\beta_0}}\right)\right)^{\frac{1}{2}} dx\end{aligned} \tag{4.11}$$

Since $\|u_n\|^2 = \frac{2\pi(2-a)}{\beta_0} - \delta$, it is clear that the function $m(u_n) := e^{\beta_0 \overline{u}_n^2} - 1$ satisfies the conditions of Lemma 2.4 , so the first term tends to zero. By Lebesgues dominated convergence, we can see also that the second term tends to zero.

From H_5) and Lemma 2.4, we can estimate the third term by

$$\begin{aligned}\int_\Omega \frac{g(v_n)}{\sqrt{\beta_0}\,|x|^a}\left(\log\left(\frac{g(v_n)}{\sqrt{\beta_0}}\right)\right)^2 dx &\leq \int_\Omega \frac{g(v_n)}{\sqrt{\beta_0}\,|x|^a}\left(\log\left(\frac{C_\epsilon e^{(\beta_0+\epsilon)v_n^2}}{\sqrt{\beta_0}}\right)\right)^{\frac{1}{2}} dx \\ &\leq \int_\Omega \frac{g(v_n)}{\sqrt{\beta_0}\,|x|^a}\left(\log\left(\frac{C_\epsilon}{\sqrt{\beta_0}}\right)^{\frac{1}{2}} + (\beta_0+\epsilon)^{\frac{1}{2}} v_n\right) dx \\ &\leq o(1) + \left(1+\frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}} \int_\Omega \frac{g(v_n)\,v_n}{|x|^a},\end{aligned}$$

and hence, by (4.11), we get

$$\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \|u_n\| \leq o(1) + \left(1+\frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}} \int_\Omega \frac{g(v_n)\,v_n}{|x|^a} dx \qquad (4.12)$$

Similarly, with $\|v_n\|^2 \leq \int_\Omega \frac{f(u_n)v_n}{|x|^a} dx$, we get

$$\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \|v_n\| \leq o(1) + \left(1+\frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}} \int_\Omega \frac{f(u_n)\,u_n}{|x|^a} dx \qquad (4.13)$$

On the other hand, by Lemma 2.5 and (4.7), we can conclude that

$$\int_\Omega \frac{F(u_n)}{|x|^a} dx \to 0, \quad \int_\Omega \frac{G(v_n)}{|x|^a} dx \to 0 \qquad (4.14)$$

and

$$\left|\int_\Omega \nabla u_n \nabla v_n dx\right| \leq o(1) + \frac{2\pi(2-a)}{\beta_0} - \delta,$$

which, together with (4.8), imply that

$$\int_\Omega \frac{f(u_n)\,u_n}{|x|^a} dx + \int_\Omega \frac{g(v_n)\,v_n}{|x|^a} dx \leq o(1) + 2\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)$$

So, from (4.12) and (4.13) we obtain

$$\begin{aligned}\|u_n\| + \|v_n\| &\leq o(1) + 2\left(1+\frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}}\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \\ &\leq 2\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}},\end{aligned}$$

for ϵ sufficiently small and n sufficiently large. It follows that there is a subsequence of (u_n) or (v_n) (without loss of generality assume it is (v_n)) such that

$$\|v_n\| \leq \left(\frac{2\pi\,(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}}.$$

Thus, using H_5), Lemma 2.1, and Hölder inequality with $q > 1$ such that $q\left(\frac{(\beta_0+\epsilon)\left(\frac{2\pi(2-a)}{\beta_0}-\delta\right)}{4\pi} + \frac{a}{2}\right) \leq 1$, we get

$$\begin{aligned}\left|\int_\Omega \frac{g\,(v_n)\,v_n}{|x|^a}dx\right| &\leq C_\epsilon \left\|v_n\right\|_{L^{q'}(\Omega)} \int_\Omega \frac{e^{q(\beta_0+\epsilon)v_n^2}}{|x|^{qa}}dx \\ &\leq C \left\|v_n\right\|_{L^{q'}(\Omega)}.\end{aligned}$$

Since $\|v_n\|_{L^{q'}(\Omega)} \to 0$, we get

$$\int_\Omega \frac{g\,(v_n)\,v_n}{|x|^a}dx \to 0.$$

Hence,

$$\int_\Omega \nabla u_n \nabla v_n dx \to 0 \tag{4.15}$$

which, together with (4.14), imply that $c_n \to 0$, yielding a contradiction.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press (1975).

[2] Adimurthi, K. Sandeep, *A singular Moser-Trudinger embedding and its applications*, NoDEA Nonlinear Differential Equations Appl. 13 (2007) 585-603.

[3] Q. Dai, and Y. Gu, *Positive solutions for non-homogeneous semilinear elliptic equations with data that changes sign*, Proc. Roy. Soc. Edin. 133A (2003), 297-306

[4] L. Fang, Y. Jianfu, *Nontrivial solutions of Hardy-Hénon type elliptic systems*, Acta Mathematica Scientia 27 (2007), 673-688.

[5] D. G. De Figuerido, P. Felmer, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. 343 (1994), 99-116.

[6] D. G. De Figuerido, O. H. Miyagaki, B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. PDE. 3 (1995) 139-153.

[7] D. G. De Figuerido, J. M. do Ó, B. Ruf, *Critical and subcritical elliptic systems in demension two*, Indiana University Math. J. 53 (2005) 1037-1054.

[8] D. G. De Figuerido, I. Peral, J. D. Rossi, *The critical hyperbola for a Hamiltonian elliptic system with weights*, Annali di Matematica 187 (2008), 531-545.

[9] D. G. De Figuerido, B. Ruf, *Elliptic systems with nonlinearities of arbitrary growth*, Mediterr. J. Math. 1 (2004), 417-431.

[10] P. Han, *Strongly indefinite systems with critical Sobolev exponents and weight*, Applied Mathematics letters, 17 (2004) 909-917.

[11] J. Hulshof, E. Mitidieri, R.C.A.M. Van der Vorst, *Strongly indefinite systems with critical Sobolev exponents* Trans. Amer. Math. Soc. 350 (1998), 2349-2365.

[12] N. Megrez, K. Sreenadh, B. Khaldi, *Multiplicity of positive solutions for an elliptic gradient system with exponential nonlinearity.* Electronic Journal of Differential Equations, No. 236 (2012), 1-16.

[13] J. Moser, *A sharp form of an inequality by N.Trudinger*, Indiana Univ. Math. Jour. 20 (1971), 1077-1092.

[14] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math. 65, AMS, Providence, RI, 1986.

[15] B. Ruf, *Lorentz spaces and nonlinear elliptic systems,* Progr. Nonlin. Diff. Equ. 66 (2005), 471-489.

[16] M. de Souza, J. M. do Ó, *On a singular and nonhomogeneous N-Laplacian equation involving critical growth*, J. Math. Anal. Appl. 280 (2011) 241-263.

[17] M. de Souza, *On a singular class of elliptic systems involving critical growth in* $\mathbb{R}^2$, Nonl. Anal. Real World Appl. 12 (2011), 1072-1088

[18] N. S. Trudinger, *On embedding into Orlicz spaces some applications*, J. Math. Mech. 17 (1967), 473-484.

[19] M. de Souza, *On a singular Hamiltonian elliptic systems involving critical growth in dimension two*, Communications on Pure and Applied Analysis, 11 (2012), 1859-1874.