American. Jr. of Mathematics and Sciences
Vol. 1, No.1 ,(January 2012)

www.journalshub.com

GENERALIZED ENTROPIES VIA FUNCTIONAL EQUATIONS AND DETERMINING MEAN CODEWORD LENGTHS

Om Parkash
Department of Mathematics
Guru Nanak Dev University
Amritsar- 143005 (India)

Priyanka Kakkar
Department of Mathematics
Guru Nanak Dev University
Amritsar-143005 (India)

ABSTRACT

Usually in the literature of information theoretic measures, entropies are expressed as the weighted arithmetic mean of their generating functions, the findings of which are sometimes pseudo-additive entropies. In the present communication, we have expressed the generalized existing entropies in terms of mean values and determined the parametric mean codeword lengths using the additivity condition. Power mean and geometric mean have been determined using the concept of multiplicativity of means and it is shown that that multiplicative means serves as a lower and upper bound to the newly introduced additive mean codeword length.

Keywords: ***Functional equations, Entropy, Mean codeword length, Convex function, Kraft's*** **inequality, Monotonicity.**

INTRODUCTION

The concept of information entropy introduced by well known mathematician Shannon [14] gave birth to many entropies making the literature on information theory voluminous. These entropies are famous as parametric, trigonometric and weighted entropies. The entropy measure introduced by Shannon [14] is given by the following mathematical expression:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.1}$$

A systematic attempt to develop a generalization of Shannon's [14] entropy was carried out by Renyi [13], who characterized an entropy of order α given by

$$H_\alpha(P) = \frac{1}{1-\alpha}\log\left(\sum_{i=1}^{n} p_i^\alpha\right), \quad \alpha \neq 1, \alpha > 0 \tag{1.2}$$

Based upon Renyi's [13] motivations, Aczel and Daroczy [2], Kapur [7] generalized the entropy of order α by changing some of its postulates. The generalization provided by Aczel and Daroczy [2], known as entropy of order β is given by

$$_\beta H_n(p_1, p_2, \ldots, p_n) = -\sum_{k=1}^{n} \frac{p_k^\beta \log p_k}{\sum_{k=1}^{n} p_k^\beta}, \quad \beta > 0 \tag{1.3}$$

whereas Kapur [7] introduced entropy of order α and type β given by

$$_{\alpha,\beta} H_n(p_1, p_2, \ldots, p_n) = \frac{1}{1-\alpha}\log\left(\sum_{k=1}^{n} \frac{p_k^{\alpha+\beta-1}}{\sum_{k=1}^{n} p_k^\beta}\right), \alpha > 0, \ \alpha \neq 1, \ \beta > 0 \tag{1.4}$$

It is well known that Kraft's [8] inequality play an important role in proving noiseless coding theorems. Nagaraja [10] remarked that Kraft–McMillan inequality is a basic result in information theory which gives a necessary and sufficient condition for a code to be uniquely decodable and also has a quantum analogue. The author proved this inequality and its converse for prefix-free codes. Ludwig [9] remarked that Kraft's inequality is a classical theorem in Information Theory which establishes the existence of prefix codes and proved a generalization of this inequality which states that for every admissible infinite length distribution, one can construct a maximal prefix codes whose codewords satisfy this length distribution. Some other findings related with the construction of codeword lengths have been given by Baer [3], Ramamoorthy [12], Dar, R. A. and Baig, M. A. K. [5] etc.

The axiomatic characterizations of various entropies have little apparent connection with the mean code-word lengths. In this paper, we have defined generalized entropy in such a way that connects entropy directly with the mean code-word lengths. In the next section, we have characterized the existing measures of entropy introduced by Aczel and Daroczy [2] and Kapur [7] by applying functional equations.

2. DEVELOPMENT OF GENERALIZED ENTROPIES VIA FUNCTIONAL EQUATIONS

We define a parametric class, denoted by F^* as follows:

1. A function $\phi :]0,1] \to R$ belongs to the class F^* if ϕ is strictly monotonic and the function $\phi^* : [0,1] \to R$ defined by

$$\phi^*(t) = \begin{cases} \dfrac{t^\beta \phi(t)}{\sum t^\beta}, \ t \in]0,1], \quad \beta > 0 \\ 0 \quad , \quad t = 0 \end{cases} \tag{2.1}$$

is continuous on [0,1].

2. Let $\phi \in F^*$. Generalized mean value is defined by

$$ {}^\phi M_n(p_1, p_2, \ldots, p_n) = \phi^{-1}\left[\sum_{i=1}^{n} \phi^*(p_k)\right] \tag{2.2}$$ where

ϕ^* is defined by (2.1)

3. Let $\phi \in F^*$ as defined in def 1and let $\{{}^\phi M_n\}$ be as defined in definition 2. Then

$$ {}^\phi I_n(p_1, p_2, \ldots, p_n) = -\log \phi^{-1}\left[\sum_{i=1}^{n} \phi^*(p_k)\right] \tag{2.3}$$

are called generalized $\phi-$entropies. This definition is however, too wide to yield an entropy concept useful for applications. Thus, we impose two restrictions, first with the aid of class F^* defined in definition 1 and second as the condition of additivity, that is, generalized $\phi-$entropies should be additive. We observe that the additive property of generalized $\phi-$entropies, is translated, according to (2.3) into the multiplicative property of generalized mean.

Concept of equality of generalized mean

The two functions $\phi, \Phi \in F^*$, the $\phi-$generalized mean and the $\Phi-$generalized mean are equal,

that is,
$$\phi^{-1}\left[\sum_{k=1}^{n} \frac{p_k^\beta}{\sum_{k=1}^{n} p_k^\beta} \phi(p_k)\right] = \Phi^{-1}\left[\sum_{k=1}^{n} \frac{p_k^\beta}{\sum_{k=1}^{n} p_k^\beta} \Phi(p_k)\right] \tag{2.4}$$

iff ϕ and Φ are affine maps of each other.

Theorem 2.1: Let $\phi, \Phi \in F^*$, then (2.4) is satisfied iff $\exists$ constants $A \neq 0$ and B such that

$$\Phi(p) = A\phi(p) + B \quad \forall \ p \in]0,1] \tag{2.5}$$

Proof: One sees immediately that (2.4) is satisfied if (2.5) is satisfied. We have to show that (2.5) implies (2.4).Define the continuous function h by

$$h(x) = \Phi[\phi^{-1}(x)] \quad \forall \ x \in]0,1] \tag{2.6}$$

Then (2.4) gives $h\left[\sum_{k=1}^{n}\frac{p_k^{\beta}}{\sum_{k=1}^{n}p_k^{\beta}}\phi(p_k)\right]=\sum_{k=1}^{n}\frac{p_k^{\beta}}{\sum_{k=1}^{n}p_k^{\beta}}h(\phi(p_k))$

Then, there exist constants A and B such that $h(x)=Ax+B$ for all $x\in\phi(]0,1])$ (refer Aczel [1]).
By comparing this equation with (2.6), we have

$$\Phi(p)=A\phi(p)+B \quad \forall\ p\in\,]0,1]$$

The strict monotonicity of Φ implies $A\neq 0$ and this concludes the proof of the theorem.

Theorem 2.2: Let $\phi\in F^{*}$, then the generalized entropy $\{{}^{\phi}I_n\}$ is additive iff

$$\phi(t)=\log t \tag{2.7}$$

or

$$\phi(t)=t^{\alpha-1} \tag{2.8}$$

for all $t\in\,]0,1]$ hold, upto an additive and non-zero multiplicative constant α. This means that among the generalized entropies, the only entropies additive in nature are given by

$$_{\beta}H_n(p_1,p_2,...,p_n)=-\sum_{k=1}^{n}\frac{p_k^{\beta}\log p_k}{\sum_{k=1}^{n}p_k^{\beta}},\qquad \beta>0 \tag{2.9}$$

and

$$_{\alpha,\beta}H_n(p_1,p_2,...,p_n)=\frac{1}{1-\alpha}\log\left(\sum_{k=1}^{n}\frac{p_k^{\alpha+\beta-1}}{\sum_{k=1}^{n}p_k^{\beta}}\right),\ \beta>0 \tag{2.10}$$

where (2.9) and (2.10) are respectively Aczel and Darcozy [2] and Kapur's [6] entropy.

Proof: The "only if" part is easily checked. As to the "if" statement, let $\phi\in F^{*}$ and the generalized $\phi-$entropy ${}^{\phi}I_n$ be additive. Then, the respective $\phi-$generalized mean is multiplicative, that is,

$$\phi^{-1}\left[\sum_{k=1}^{n}\sum_{j=1}^{m}\phi^{*}(p_kq_j)\right]=\phi^{-1}\left[\sum_{k=1}^{n}\phi^{*}(p_k)\right]\phi^{-1}\left[\sum_{j=1}^{m}\phi^{*}(q_j)\right] \tag{2.11}$$

Again, we will suppose this only for non-zero probabilities, that is,

$$\phi^{-1}\left[\sum_{k=1}^{n}\sum_{j=1}^{m}\frac{p_k^{\beta}q_j^{\beta}}{\sum_{k=1}^{n}\sum_{j=1}^{m}p_k^{\beta}q_j^{\beta}}\phi(p_kq_j)\right]=\phi^{-1}\left[\sum_{k=1}^{n}\frac{p_k^{\beta}}{\sum_{k=1}^{n}p_k^{\beta}}\phi(p_k)\right]\phi^{-1}\left[\sum_{j=1}^{m}\frac{q_j^{\beta}}{\sum_{j=1}^{m}q_j^{\beta}}\phi(q_j)\right]$$

(2.12)

If (2.12) is satisfied, then, by (2.1), (2.11) also holds.

Now, put $q_j=\frac{1}{m}(j=1,2,\ldots,m)$ into (2.12), we get

$$\phi^{-1}\left[\sum_{k=1}^{n}\frac{p_k^{\beta}}{\sum_{k=1}^{n}p_k^{\beta}}\phi\left(\frac{p_k}{m}\right)\right]=\phi^{-1}\left[\sum_{k=1}^{n}\frac{p_k^{\beta}}{\sum_{k=1}^{n}p_k^{\beta}}\phi(p_k)\right]\frac{1}{m} \tag{2.13}$$

If, for a fixed m, we denote

$$\Phi(x):=\phi\left(\frac{x}{m}\right) \tag{2.14}$$

then (2.13) becomes (2.4) and $\Phi\in F^{*}$, and $\phi\in F^{*}$. So, by Theorem (2.1), we have

$$\Phi(x) = A\phi(x) + B \quad \forall\ x \in]0,1] \tag{2.15}$$

for a fixed m. If m goes through the positive integers, the constants A and B can be different for different m and so (2.15) really means

$$\phi\left(\frac{x}{m}\right) = A(m)\phi(x) + B(m) \quad \forall\ x \in]0,1], \quad m = 1,2,... \tag{2.16}$$

Our task is to find all functions ϕ, defined, continuous and strictly monotonic on $]0,1]$ which satisfy (2.16). With $x = \frac{1}{t}$ and

$$f(t) := \phi\left(\frac{1}{t}\right)\ ,\ t \in [1,\infty[\tag{2.17}$$

(2.16) becomes

$$f(tm) = A(m) f(t) + B(m) \tag{2.18}$$

By (2.17), $f : [1,\infty[\to R$ is continuous and strictly monotonic and it is known that the general solution of (2.18) is given by

$f(t) = c\log t + b$ or $f(t) = a t^c + b$, $(t \in [1,\infty[)$, $a \neq 0, c \neq 0, b$ are constants (refer Aczel [1])

Taking (2.17) into consideration in both cases, we have

$\phi(x) = -c\log x + b \quad (c \neq 0)$ and $\phi(x) = a x^{-c} + b \quad (a \neq 0,\ c \neq 0)$ for all $x \in]0,1]$, respectively, or, with other notations for the constants,

$$\phi(x) = a\log x + b \quad (a \neq 0) \tag{2.19}$$

and

$$\phi(x) = a x^{\alpha-1} + b \quad (a \neq 0,\ \alpha \neq 1) \tag{2.20}$$

However, by supposition, $\phi \in F^*$ and so (2.1) has to hold.

In particular, $\lim\limits_{x_i \to 0}\left[\frac{x_i^{\beta}}{\sum x_i^{\beta}}\phi(x_i)\right] = 0$. Thus, from (2.20), we have

$$0 = \lim_{x_i \to 0^+}\left[\frac{x_i^{\beta}}{\sum x_i^{\beta}}\phi(x_i)\right] = \lim_{x_i \to 0^+}\left[\frac{x_i^{\beta}}{\sum x_i^{\beta}}\left(a x_i^{\alpha-1} + b\right)\right] = \lim_{x_i \to 0^+}\left[a\,\frac{x_i^{\alpha+\beta-1}}{\sum x_i^{\beta}} + b\,\frac{x_i^{\beta}}{\sum x_i^{\beta}}\right]$$

Hence, we have to have $\alpha > 0$. So, (2.19) and (2.20) show that (2.7) and (2.8), with arbitrary $\alpha > 0,\ \alpha \neq 1$, are indeed, upto an additive and a non-zero multiplicative constant, the most general solutions in F^*. If we put these ϕ into the definition of the generalized $\phi-$entropies, we get Aczel and Darcozy [2] and Kapur's [6] entropy, respectively.

In next sections, we have characterized generalized parametric mean codeword lengths through additive and multiplicative conditions and consequently, developed some desirable inequalities.

3. DETERMINATION OF MEAN CODEWORD LENGTHS AND THEIR RELATED INEQUALITIES

Let $P = \left\{\frac{p_1^{\beta}}{\sum_{i=1}^{m} p_i^{\beta}}, \frac{p_2^{\beta}}{\sum_{i=1}^{m} p_i^{\beta}}, \ldots, \frac{p_m^{\beta}}{\sum_{i=1}^{m} p_i^{\beta}}\right\}$, $\beta > 0$ be a set of complete probability distribution. Suppose that we wish to represent the events in X by finite sequences of elements of the set $\{0,1,...,D-1\}$ where $D > 1$. There is a uniquely decipherable code which represents x_i by a sequence of n_i elements if and only if the integers n_i satisfy the Kraft's [8] inequality

$$\sum_{i=1}^{m} D^{-n_i} \leq 1 \tag{3.1}$$

Let $\psi : [1,\infty[\to R$ be continuous and strictly monotonic function known as "cost function" associated with the length so that the "cost" of using a sequence of length n is $\psi(n)$. Then, the average cost of encoding X by a distribution of lengths $N = \{n_1, n_2, ..., n_m\}$ is given by

$$C = \sum_{i=1}^{m} \frac{p_i^{\beta}}{\sum_{i=1}^{m} p_i^{\beta}} \psi(n_i).$$

Since ψ is continuous and strictly monotonic function on $[1,\infty[$, therefore ψ has an inverse ψ^{-1}. We can now define a mean length for the cost function ψ by

$$L(P, N, \psi) = \psi^{-1}(C) = \psi^{-1}\left(\sum_{i=1}^{m} \frac{p_i^{\beta}}{\sum_{i=1}^{m} p_i^{\beta}} \psi(n_i)\right),\ \beta > 0 \tag{3.2}$$

The reason for calling L a mean length is that when $n_1 = n_2 = ... = n_m = n$, then $L = n$.

Moreover, if $\psi(x) = \psi_0(x) = x,$ then

$$L(P, N, \psi_0) = \frac{\sum_{i=1}^{m} p_i^{\beta} n_i}{\sum_{i=1}^{m} p_i^{\beta}}, \tag{3.3}$$

a mean codeword length developed by Parkash and Priyanka [11]. Also, if $\psi_0(x) = x,\ \beta = 1$, then

$$L(P, N, \psi_0) = \sum_{i=1}^{m} p_i n_i, \tag{3.4}$$

which is an ordinary mean length.

Kapur [7] has also introduced the 2-parameter exponentiated mean codeword length of order $t, t > 0$ and type β for which $\psi(x) = \psi_t(x) = D^{tx}$, $x \in [1,\infty[$ given by

$$L(P, N, \psi_t) = \frac{1}{t} \log_D \left(\sum_{i=1}^{m} \frac{D^{tn_i} p_i^{\beta}}{\sum_{i=1}^{m} p_i^{\beta}} \right) \tag{3.5}$$

Also, if $\psi(x) = \psi_t(x) = D^{tx}$, $\beta = 1$ then (3.5) becomes

$$L(P, N, \psi_t) = \frac{1}{t} \log_D \left(\sum_{i=1}^{m} D^{tn_i} p_i \right) \tag{3.6}$$

which is a exponential mean codeword length introduced by Campbell [4].

Important inequalities have been developed by Shannon [14] and Campbell [4] for the mean codeword lengths (3.4) and (3.6). These give essentially Shannon [14] and Renyi [13] entropies as lower bounds of (3.4) and (3.6). Next, we determine the generalized mean codeword lengths:

3.1 Determination of generalized additive mean codeword lengths

Here we will show that how the natural additivity condition characterizes the mean codeword lengths (3.3) and (3.5). Consider two independent set of events $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_r\}$ with the probability distributions $P=\left\{\frac{p_1^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}},\frac{p_2^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}},\ldots,\frac{p_m^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}\right\}$ and $Q=\left\{\frac{q_1^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}},\frac{q_2^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}},\ldots,\frac{q_r^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}}\right\}$ respectively. Since X and Y are independent, the probability of pair (x_i,y_j) is $\frac{p_i}{\sum_{i=1}^{m}p_i^{\beta}}\frac{q_j}{\sum_{j=1}^{r}q_j^{\beta}}$. Let PQ denote the set of probabilities $\frac{p_i}{\sum_{i=1}^{m}p_i^{\beta}}\frac{q_j}{\sum_{j=1}^{r}q_j^{\beta}}$. Let A be a set of D symbols, and let $S(X,A)$ and $S(Y,A)$ be uniquely decipherable codes with the sequences $N=(n_1,n_2,\ldots,n_m)$ and $M=(m_1,m_2,\ldots,m_r)$ of lengths of code words. Then by Kraft's [8] inequality, we have $\sum_{i=1}^{m}D^{-n_i}\le 1$, $\sum_{j=1}^{r}D^{-m_j}\le 1$ and so $\sum_{i=1}^{m}\sum_{j=1}^{r}D^{-(n_i+m_j)}=\sum_{i=1}^{m}D^{-n_i}\sum_{j=1}^{r}D^{-m_j}\le 1$. Thus, there exists a code $S(X\times Y,A)$ for which the family of the lengths of code words is exactly $N+M=\{n_i+m_j : i=1,2,\ldots,m;\ j=1,2,\ldots,r\}$.

Then, if L is to be a measure of mean length, it is natural to require that

$$L(PQ,N+M,\psi)=L(P,N,\psi)+L(Q,M,\psi)$$

or

$$\psi^{-1}\Big(\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{p_i^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}\frac{q_j^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}}\psi(n_i+m_j)\Big)=\psi^{-1}\Big(\sum_{i=1}^{m}\frac{p_i^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}\psi(n_i)\Big)+\psi^{-1}\Big(\sum_{j=1}^{r}\frac{q_j^{\beta}}{\sum_{j=1}^{k}q_j^{\beta}}\psi(m_j)\Big)\qquad(3.7)$$

This is known as the additivity condition.

Theorem 3.1 The new mean codeword length (3.3) and (3.5) are the only generalized mean codeword lengths which are additive with $m=r=2$.

Proof: For $m=r=2$, (3.7) can be written as

$$\psi^{-1}\Big(\frac{p_1^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\frac{q_1^{\beta}}{\sum_{j=1}^{2}q_j^{\beta}}\psi(n_1+m_1)+\frac{p_2^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\frac{q_1^{\beta}}{\sum_{j=1}^{2}q_j^{\beta}}\psi(n_2+m_1)+\frac{p_1^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\frac{q_2^{\beta}}{\sum_{j=1}^{2}q_j^{\beta}}\psi(n_1+m_2)+\frac{p_2^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\frac{q_2^{\beta}}{\sum_{j=1}^{2}q_j^{\beta}}\psi(n_2+m_2)\Big)$$

$$=\psi^{-1}\Big(\frac{p_1^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\psi(n_1)+\frac{p_2^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\psi(n_2)\Big)+\psi^{-1}\Big(\frac{q_1^{\beta}}{\sum_{j=1}^{2}q_j^{\beta}}\psi(m_1)+\frac{q_2^{\beta}}{\sum_{j=1}^{2}q_j^{\beta}}\psi(m_2)\Big)\qquad(3.8)$$

Put $m_1=m_2=a$, $\frac{p_1^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}=1-p$, $\frac{p_2^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}=p$ in (3.8) to get

$$\psi^{-1}\big((1-p)\psi(n_1+a)+p\psi(n_2+a)\big)=\psi^{-1}\big((1-p)\psi(n_1)+p\psi(n_2)\big)+a\qquad(3.9)$$

for all $p\in[0,1]$, n_1,n_2,a positive integers

But it is known that the only strictly monotonic increasing solutions of (3.9) are $\psi_0(x)=\gamma x+b,\gamma\neq 0$ and $\psi_t(x)=\gamma D^{tx}+b,\ t\neq 0,\gamma\neq 0$ (refer Aczel and Daroczy [2]).

When $\psi_0(x)=\gamma x+b$, then $L(P,N,\psi_0)=\psi^{-1}\left(\sum_{i=1}^{m}\frac{p_i^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}(\gamma n_i+b)\right)=\frac{\sum_{i=1}^{m}p_i^{\beta}n_i}{\sum_{i=1}^{m}p_i^{\beta}}$ which is (3.3).

When $\psi_t(x)=\gamma D^{tx}+b$, then $L(P,N,\psi_t)=\psi^{-1}\left(\sum_{i=1}^{m}\frac{p_i^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}(\gamma D^{tn_i}+b)\right)=\frac{1}{t}\log_D\left(\sum_{i=1}^{m}\frac{D^{tn_i}p_i^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}\right)$

which is (3.5).This completes the proof.

For $b=0,\gamma=1$, we have $\psi_t(x)=D^{tx}$ which is a convex function for $t>0$.

Using the convexity of ψ_t and monotonicity of ψ_t^{-1}, we have

$$\sum_{i=1}^{m}\frac{p_i^{\beta}n_i}{\sum_{i=1}^{m}p_i^{\beta}}\le\frac{1}{t}\log_D\left(\sum_{i=1}^{m}\frac{p_i^{\beta}D^{tn_i}}{\sum_{i=1}^{m}p_i^{\beta}}\right) \tag{3.10}$$

For $\beta=1$, (3.10) gives $\sum_{i=1}^{m}p_in_i\le\frac{1}{t}\log_D\left(\sum_{i=1}^{m}p_iD^{tn_i}\right)$.

4. MULTIPLICATIVE CHARACTERIZATION OF MEANS AND THEIR RELATION TO MEAN CODEWORD LENGTH

In the section-2, we have defined the generalized mean and their corresponding ϕ- entropies. We observe that the additive property of ϕ- entropies is translated according to (2.3) into multiplicative property of generalized mean. Using this concept in the context of coding theory, we shall characterize the means which are multiplicative in nature. Here, we extend the domain of function ϕ from $[0,1]$ to $[1,\infty[$.

Let $P=\left\{\frac{p_1^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}},\frac{p_2^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}},\ldots,\frac{p_m^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}\right\}$ and $Q=\left\{\frac{q_1^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}},\frac{q_2^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}},\ldots,\frac{q_r^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}}\right\}$, $\beta>0$ be the set of complete probability distributions associated with the independent set of events $X=\{x_1,x_2,\ldots,x_m\}$ and $y=\{y_1,y_2,\ldots,y_r\}$ respectively. Since X and Y are independent, the probability of the pair (x_i,y_j) is $\frac{p_i^{\beta}}{\sum_{i=1}^{m}p_i^{\beta}}\frac{q_j^{\beta}}{\sum_{j=1}^{r}q_j^{\beta}}$. Let x_i and y_j be represented by sequences of lengths $n_i, i=1,2,..,m$ and m_j, $j=1,2,..,r$ respectively. Using multiplicative law, we have

$${}^{\phi}M_{mr}(PQ)={}^{\phi}M_m(P).{}^{\phi}M_r(Q)$$

Let $m=r=2, m_1=m_2=a$, then from the above equation, we have

$$\phi^{-1}\left(\sum_{i=1}^{2}\frac{p_i^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\phi(n_ia)\right)=\phi^{-1}\left(\sum_{i=1}^{2}\frac{p_i^{\beta}}{\sum_{i=1}^{2}p_i^{\beta}}\phi(n_i)\right).a \tag{4.1}$$

But it is known that the only strictly monotonic increasing solutions of (4.1) are $\phi_0(x) = \gamma \log x + \alpha$ and $\phi_1(x) = \gamma x^c + \alpha, \quad x > 0,\ c \neq 0, \gamma \neq 0$ (refer Aczel [1])

Taking $\phi_0(x) = \gamma \log x + \alpha$, we have $\;{}^{\phi_0}M_m(P) = \phi^{-1}\Big(\sum_{i=1}^{m} \frac{p_i^\beta}{\sum_{i=1}^{m} p_i^\beta}(\gamma \log n_i + \alpha)\Big) = \prod_{i=1}^{m} n_i^{\frac{p_i^\beta}{\sum_{i=1}^{m} p_i^\beta}}$

which is geometric mean codeword length.

Next taking, $\phi_1(x) = \gamma x^c + \alpha$, we have ${}^{\phi_1}M_m(P) = \phi^{-1}\Big(\sum_{i=1}^{m} \frac{p_i^\beta}{\sum_{i=1}^{m} p_i^\beta}(\gamma n_i^{\,c} + \alpha)\Big) == \left(\sum_{i=1}^{m} \frac{p_i^\beta}{\sum_{i=1}^{m} p_i^\beta} n_i^{\,c}\right)^{\frac{1}{c}}$

which is a power mean of order c. Next, we consider the following cases:

Case-I: When $\alpha = 0, \gamma = 1$, we have $\phi_0(x) = \log x, x \in N$ which is concave function of x.

Using the concavity of ϕ_0 and monotonicity of $\phi_0^{\,-1}$, we have

$$M \leq \sum_{i=1}^{m} \frac{p_i^\beta n_i}{\sum_{i=1}^{m} p_i^\beta} \tag{4.2}$$

where $M = \prod_{i=1}^{m} n_i^{\frac{p_i^\beta}{\sum_{i=1}^{m} p_i^\beta}}$ denotes the geometric mean of sequences of lengths $n_i, i = 1,2,..,m$.

Case-II: When $\alpha = 0, \gamma = 1$, we have $\phi_1(x) = x^c$ which is a convex function of x.

Using the convexity of ϕ_1 and monotonicity of $\phi_1^{\,-1}$, we have

$$\left(\sum_{i=1}^{m} \frac{p_i^\beta n_i}{\sum_{i=1}^{m} p_i^\beta}\right) \leq \left(\sum_{i=1}^{m} \frac{p_i^\beta n_i^c}{\sum_{i=1}^{m} p_i^\beta}\right)^{\frac{1}{c}} \tag{4.3}$$

So, from (4.2) and (4.3), we have

$$\prod_{i=1}^{m} n_i^{\frac{p_i^\beta}{\sum_{i=1}^{m} p_i^\beta}} \leq \left(\sum_{i=1}^{m} \frac{p_i^\beta n_i}{\sum_{i=1}^{m} p_i^\beta}\right) \leq \left(\sum_{i=1}^{m} \frac{p_i^\beta n_i^c}{\sum_{i=1}^{m} p_i^\beta}\right)^{\frac{1}{c}} \tag{4.4}$$

which shows that the additive mean codeword length is bounded above and below by multiplicative mean codeword lengths. In particular, for $\beta = 1$, (4.4) gives

$$\prod_{i=1}^{m} n_i^{\,p_i} \leq \left(\sum_{i=1}^{m} p_i n_i\right) \leq \left(\sum_{i=1}^{m} p_i n_i^c\right)^{\frac{1}{c}}$$

which shows that arithmetic mean codeword length is bounded above and below by power mean and geometric mean, respectively.

ACKNOWLEDGEMENTS: The authors are thankful to University Grants Commission and Council of Scientific and Industrial Research, New Delhi, for providing the financial assistance for the preparation of the manuscript.

REFERENCES

[1] Aczel, J. (1966): "Functional Equations and their Applications", Academic Press, New York.

[2] Aczel, J. and Daroczy, Z. (1975): "On Measures of Information and their Characterizations", Academic Press.

[3] Baer, M. B. (2011): "Redundancy-related bounds for generalized Huffman codes", IEEE Trans. Inform. Theory", **57**(4), 2278-2290.

[4] Campbell, L.L. (1965): "A coding theorem and Renyi's entropy", Inform and Control, **8**, 423-429.

[5] Dar, R. A. and Baig, M. A. K. (2008): "Coding theorems on generalized cost measures", Journal of inequalities in pure and applied mathematics, **9**, 8(1)-8(6).

[6] Kapur, J.N. (1967): "Generalized entropy of order α and type β", The Math. Seminar**,** **4**, 78-94.

[7] Kapur, J.N. (1998): "Entropy and Coding", MSTS.

[8] Kraft, L.G. (1949): "A Device for Quantizing, Grouping, and Coding Amplitude Modulated Pulses", MS Thesis, Electrical Engineering Department, Massachusetts Institute of Technology.

[9] Ludwig, S. (2007): "On maximal prefix codes", Bull. Eur. Assoc. Theor. Comput. Sci., **91**, 205–207.

[10] Nagaraja, N. (2009). "A dynamical systems proof of Kraft–McMillan inequality and its converse for prefix-free codes", Chaos, **19**, 013136(1)-013136(5).

[11] Parkash, O. and Kakkar, P. (2011): "Development of two new mean codeword lengths" (Revised).

[12] Ramamoorthy, A. (2011): "Minimum cost distributed source coding over a network", IEEE Trans. Inform. Theory, **57**(1), 461-475.

[13] Renyi, A. (1971): "On measures of entropy and information", Proc. Fourth Berkeley Symposium Math. Statis. and Prob., **1** ,547-561.

[14] Shannon, C.E. (1948): "A mathematical theory of communication," Bell Syst. Tech. J., **27**, 379–423 (Part I), 623–656 (Part II).