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On a critical and subcritical system of subelliptic equations on unbouded domain of Heisenberg group

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Abstract

We give two existence results for the problem

$$(P): \begin{cases} -\Delta_{\mathbb{H}^n} u = q|v|^{q-2}v \text{ in } \Omega \\ -\Delta_{\mathbb{H}^n} v = p|u|^{p-2}u \text{ in } \Omega \\ \lim\limits_{|\xi|\longrightarrow\infty} u(\xi) = 0 \\ \lim\limits_{|\xi|\longrightarrow\infty} v(\xi) = 0 \\ u_{|\partial\Omega} = v_{|\partial\Omega} = 0 \text{ (if } \Omega \neq \mathbb{H}^n) \end{cases}$$

where $\Delta_{\mathbb{H}^n}$ is the Heisenberg Laplacian and $\mathbb{H}^n$ is the Heisenberg group. The first existence result is established when Ω is a strongly asymptoticaly contractive domain, and $p,\ q \leq 2\frac{n+1}{n-1}$ are superlinear-subcritical, that is $1 > \dfrac{1}{p} + \dfrac{1}{q} > \dfrac{n}{n+1}$. The second existence result is established when $\Omega = \mathbb{H}^n$, and (p, q) is critical, that is $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{n}{n+1}$.

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1 Introduction and main result

We denote by $\mathbb{H}^n$ the vector space $\mathbb{R}^{2n+1}$, of vectors $\xi := (x_1, ..., x_n, y_1, ..., y_n, t) := (x, y, t)$, endowed with the group action:

$$\xi \circ \xi_0 = (x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i})).$$

$\mathbb{H}^n$ is a Lie group, called the Heisenberg group, and the corresponding Lie algebra of left invariant vector fields, is generated by:

$$\begin{cases} X_i = \dfrac{\partial}{\partial x_i} + 2y_i \dfrac{\partial}{\partial t}\ ,\ i = 1, ..., n, \\ Y_i = \dfrac{\partial}{\partial y_i} - 2x_i \dfrac{\partial}{\partial t}\ ,\ i = 1, ..., n, \\ T = \dfrac{\partial}{\partial t}. \end{cases}$$

We have $[X_i, Y_j] = -4T\delta_{i,j},\ [X_j, X_k] = [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0.$
The Heisenberg Laplacian, (also called the subelliptic Laplacian, or the Kohn Laplacian), is defined as:

$$\begin{aligned} \Delta_{\mathbb{H}^n} &:= \sum_{i=1}^{n}(X_i^2 + Y_i^2) \\ &= \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} + \frac{\partial^2}{\partial {y_i}^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial t^2} \\ &= div(A\nabla u) \end{aligned}$$

where A is the following $(2n + 1) \times (2n + 1)$ matrix:

$$\begin{pmatrix} I_n & 0 & 2y^t \\ 0 & I_n & -2x^t \\ 2y & -2x & 4(x^2 + y^2) \end{pmatrix}$$

Observe that A is a positive semi definite matrix, with $det(A) \equiv 0$ for all $(x, y, t) \in \mathbb{H}^n$, and $rank(A) = 2n$.
A natural group of dilations on $\mathbb{H}^n$, is given by:

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t),\ \ \lambda > 0.$$

The Jacobian determinant of δ_λ is λ^N, where $N = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.
$N^* := \dfrac{2N}{N-2}$, is the critical Sobolev exponent for $\Delta_{\mathbb{H}^n}$.
Let Ω be an open set of $\mathbb{H}^n$. We denote by $\overset{\circ}{S_1^2}(\Omega)$ the Folland-Stein Sobolev space, defined as the closure of $C_0^\infty(\Omega)$, under the norm:

$$\|u\|^2_{S_1^2(\Omega)} := \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi.$$

Note that $\overset{\circ}{S_1^2}(\mathbb{H}^n) = S_1^2(\mathbb{H}^n)$.

Definition 1.1. A domain $\Omega \subset \mathbb{H}^n$ is said to be strongly asymptotically contractive (S.A.C for simplicity), if $\Omega \neq \mathbb{H}^n$ and for any sequence $\eta_j \in \mathbb{H}^n$ such that $|\eta_j| \longrightarrow \infty$, there exists a subsequence η_{j_l} such that either

i) $\left|\bigcup_{n=1}^{\infty}\bigcap_{l=n}^{\infty}(\eta_{j_l} \circ \Omega)\right| = 0,$

or

ii) $\exists \eta_0 \in \mathbb{H}^n$ such that for any $R > 0$ there exist an open set $M_R \subset\subset \eta_0 \circ \Omega$, a closed set Z of measure zero and an integer $l_R > 0$ such that

$$(\eta_{j_l} \circ \Omega) \cap B_R(0) \subset M_R \cup Z, \quad \text{for any } l \geq l_R.$$

Let $\Omega \subseteq \mathbb{H}^n$ be unbounded, and p, q two positives real.

In this paper, we give two existence results for the problem

$$(P): \begin{cases} -\Delta_{\mathbb{H}^n} u = q|v|^{q-2}v \text{ in } \Omega \\ -\Delta_{\mathbb{H}^n} v = p|u|^{p-2}u \text{ in } \Omega \\ \lim\limits_{|\xi| \longrightarrow \infty} u(\xi) = 0 \\ \lim\limits_{|\xi| \longrightarrow \infty} v(\xi) = 0 \\ u_{|\partial\Omega} = v_{|\partial\Omega} = 0 \text{ (if } \Omega \neq \mathbb{H}^n) \end{cases}$$

according to the following cases:

Case I: Ω is strongly asymptoticaly contractive domain, and $p,\ q \leq \dfrac{2N}{N-4}$, and are superlinear-subcritical, that is

$$1 > \frac{1}{p} + \frac{1}{q} > \frac{N-2}{N} = \frac{n}{n+1} \tag{1.1}$$

Case II: $\Omega = \mathbb{H}^n$, and (p, q) is critical, that is

$$\frac{1}{p} + \frac{1}{q} = \frac{N-2}{N} = \frac{n}{n+1} \tag{1.2}$$

Let $\mathcal{H}$ be the Banach space $\big(S_1^2(\mathbb{H}^n) \cap L^p(\mathbb{H}^n)\big) \times \big(S_1^2(\mathbb{H}^n) \cap L^q(\mathbb{H}^n)\big)$, equipped with the norm:

$$\|(u, v)\|_{\mathcal{H}} = \|u\|_{S_1^2(\mathbb{H}^n)} + \|v\|_{S_1^2(\mathbb{H}^n)}.$$

A weak solution of the problem (P) is a critical point of the functional J defined by:

$$J(u,v) := \int_\Omega \nabla_{\mathbb{H}^n} u.\nabla_{\mathbb{H}^n} v d\xi - \int_\Omega [|u|^p + |v|^q]\, d\xi \tag{1.3}$$

where $\nabla_{\mathbb{H}^n} := (X_1, ..., X_n, Y_1, ..., Y_n)$

The quadratic part of J is well defined on $\overset{\circ}{S_1^2}(\Omega)\times \overset{\circ}{S_1^2}(\Omega)$, and the second part is well defined if $p,\ q \leq \dfrac{N+2}{N-2}$. However, for subcritical (p,q) ((1.1)), at less one of the variables $p,\ q$ is greater then $\frac{N+2}{N-2}$. For this matter, we will use in case I, fractional Sobolev spaces

$$E_{s,t} = D\left((-\Delta_{\mathbb{H}^n})^{\frac{s}{2}}\right) \times D\left((-\Delta_{\mathbb{H}^n})^{\frac{t}{2}}\right), \quad s,\ t > 0, \quad s+t = 2,$$

allowing us to take $p > \frac{N+2}{N-2}$, if $q < \frac{N+2}{N-2}$, and to use some compact sobolev embedings. In case II, J is well defined and of class C^1 on $\mathcal{H}$, and in vertue of abstract concentration compactness we dont need compact embedings.

Our main results are:

Theorem 1.2. *In the case I, the problem (P) has a weak solution in $E_{s,t}$*

Theorem 1.3. *In the case II, the problem (P) has a weak solution in $\mathcal{H}$.*

2 Functional analytic frame work

In this section, we expose an abstract analytic frame work.

2.1 Spectral families

For completion, we recall results on spectral families (see [7]). Let H be a Hilbert space endowed with a scalar product $< .,. >$ and its associated norm $\|.\|$. Suppose there is a nondecreasing family $\{M(\lambda),\ \lambda \in \mathbb{R}\}$ of closed subspaces of H, such that $\bigcap_{\lambda\in\mathbb{R}} M(\lambda) = \{0\}$, and $\overline{\bigcup_{\lambda\in\mathbb{R}} M(\lambda)} = H$.
For any fixed λ, we have

$$M(\lambda - 0) := \overline{\bigcup_{\lambda'<\lambda} M(\lambda')} \subset M(\lambda) \subset M(\lambda+0) := \bigcap_{\lambda'>\lambda} M(\lambda')$$

Definition 2.1. We say that the family $\{M(\lambda)\}$ is right continuous at λ if $M(\lambda+0) = M(\lambda)$, left continuous if $M(\lambda-0) = M(\lambda)$, and continuous if it is both right and left continuous.

Definition 2.2. The family $\{E(\lambda)\}$ of orthogonal projections on $M(\lambda)$, is called spectral family, and we have:

i) $\{E(\lambda)\}$ is nondecreasing: $E(\lambda') \leq E(\lambda'')$ for $\lambda' < \lambda''$.

ii) $\lim_{\lambda \to -\infty} E(\lambda) = 0$, and $\lim_{\lambda \to +\infty} E(\lambda) = id$.

iii) $\{E(\lambda)\}$ is strongly right (resp. left) continuous if and only if $\{M(\lambda)\}$ is right (resp. left) continuous.

iv) For any semiclosed interval $I =]\lambda', \lambda''] \subset \mathbb{R}$, we define $E(I)$ as the projection on the subspace $M(I) = M(\lambda'') \ominus M(\lambda')$ [1], and we have

$$E(I) = E(\lambda'') - E(\lambda')$$

Definition 2.3. $\{E(\lambda)\}$ is said to be bounded from below if $E(\mu) = 0$ for some finite μ, that is, $\{E(\lambda)\} = 0$ for $\lambda < \mu$. The least upper bound of such μ is the lower bound of $\{E(\lambda)\}$.
$\{E(\lambda)\}$ is said to be bounded from above if $E(\mu) = id$ for some finite μ, that is, $\{E(\lambda)\} = id$ for $\lambda > \mu$. The greatest lower bound of such μ is the upper bound of $\{E(\lambda)\}$.

To any spectral family $\{E(\lambda)\}$, we associate a selfadjoint operator T defined by

$$T = \int_{-\infty}^{+\infty} \lambda dE(\lambda) \tag{2.1}$$

on

$$D(T) = \left\{ u \in H : \|Tu\|^2 = \int_{-\infty}^{+\infty} \lambda^2 d \langle E(\lambda)u, u \rangle < \infty \right\} \tag{2.2}$$

Theorem 2.4. (The spectral Theorem). *Every selfadjoint operator T admits an expression (2.1) by means of a spectral family $\{E(\lambda)\}$ which is uniquely determined by*

$$E(\lambda) = 1 - \frac{1}{2} \left[U(\lambda) + U(\lambda)^2 \right] \tag{2.3}$$

where $U(\lambda)$ is the partially isometric operator that appears in the polar decomposition $T - \lambda = U(\lambda)|T - \lambda|$, [2] *of $T - \lambda$.*

We have the following properties:

$$\langle Tu, v \rangle = \int_{-\infty}^{+\infty} \lambda d \langle E(\lambda)u, v \rangle \text{ for } u \in D(T),\ v \in H \tag{2.4}$$

$$\langle Tu, u \rangle \leq \lambda \|u\|^2 \text{ for } u \in E(\lambda)H \tag{2.5}$$

[1] $A \ominus B := A \bigcap B^{\perp}$ is the orthogonal complement of B in A.

[2] $|A| = \sum \alpha_k \langle ., \varphi_k \rangle \varphi_k$, where $\{\varphi_k\}$ is a set of eigenfunctions of A defining an orthonormal basis of H, and $\alpha_1 \geq \alpha_2 \geq \cdots > 0$ satisfie $A^* A \varphi_k = \alpha_k^2 \varphi_k,\ k = 1,\ 2\ \ldots.$

$$\lambda\|u\|^2 \leq \langle Tu, u\rangle \leq \mu\|u\|^2 \text{ for } u \in E(\mu)H \ominus E(\lambda)H \tag{2.6}$$

$$\langle Tu, u\rangle \geq \mu\|u\|^2 \text{ for } u \in [E(\mu)H]^\perp \bigcap D(T) \tag{2.7}$$

2.2 A quadratic form on fractional Sobolev spaces

Suppose that T is semi bounded from below, that is, there exists a constant δ such that

$$\langle Tu, u\rangle \geq \delta\|u\|^2, \text{ for } u \in D(T) \tag{2.8}$$

For simplicity, we will take $\delta = 1$. Then $E(\lambda) = 0$ for $\lambda < 1$, where $\{E(\lambda),\ \lambda \in \mathbb{R}\}$ denotes the spectral family associated with T. Hence one can define $T^{1/2}$ as

$$T^{1/2} := \int_1^{+\infty} \lambda^{1/2} dE(\lambda),$$

on

$$D(T^{1/2}) = \left\{u \in H :\ \|T^{1/2}u\|^2 = \int_1^{+\infty} \lambda d\left\langle E(\lambda)u, u\right\rangle < \infty\right\}.$$

For each positive real s, one can define $T^{s/2}$ as

$$A^s := T^{s/2} = \int_1^{+\infty} \lambda^{s/2} dE(\lambda),$$

on

$$E^s := D(A^s) = \left\{u \in H :\ \|A^s u\|^2 = \int_1^{+\infty} \lambda^s d\left\langle E(\lambda)u, u\right\rangle < \infty\right\}.$$

E^s is a Hilbert space, with the inner product

$$\langle u, v\rangle_{E^s} = \langle A^s u, A^s v\rangle$$

From (2.8), it follows that

$$\|A^s u\| \geq \|u\| \text{ for all } u \in E^s \tag{2.9}$$

For $s,\ t > 0$ with $s + t = 2$, we define the Hilbert space $E := E^s \times E^t$, with the inner product $\langle ., .\rangle_E = \langle ., .\rangle_{E^s} + \langle ., .\rangle_{E^t}$.
Let B be the bilinear form defined on $E \times E$ by

$$B[(u, v), (\varphi, \psi)] = \left\langle A^s u, A^t \psi\right\rangle + \left\langle A^s \varphi, A^t v\right\rangle$$

Since B is symmetric and continuous, it induces a self adjoint bounded linear operator $L :\ E \longrightarrow E$ such that

$$B[z, \eta] = \langle Lz, \eta\rangle_E, \text{ for } z,\ \eta \in E,$$

and L is defined by

$$Lz = \left\langle A^{-s}A^{t}v, A^{-t}A^{s}u\right\rangle_{E}, \text{ for } z = (u,v) \in E. \tag{2.10}$$

We consider the following eigenvalue problem

$$Lz = \lambda z \tag{2.11}$$

Using (2.10), the problem (2.11) is equivalent to

$$A^{-s}A^{t}v = \lambda u \tag{2.12}$$

and

$$A^{-t}A^{s}u = \lambda v \tag{2.13}$$

According to (2.9), A^s and A^t are isomorphisms, and then λ cannot be zero. Hence, injecting (2.12) in (2.13) we obtain

$$v = \lambda^2 v,$$

which yields that $\lambda = 1$ or $\lambda = -1$.
The corresponding eigenspaces are

$$E^{+} = \left\{(u, A^{-t}A^{s}u) : \; u \in E^{s}\right\} \text{ for } \lambda = 1,$$

and

$$E^{-} = \left\{(u, -A^{-t}A^{s}u) : \; u \in E^{s}\right\} \text{ for } \lambda = -1.$$

and are orthogonal with respect to the bilinear form B, that is,

$$B(z^{+}, z^{-}) = 0 \text{ for all } z^{+} \in E^{+}, \; z^{-} \in E^{-}.$$

We also have $E = E^{+} \oplus E^{-}$.
We define the quadratic form Q associated with the bilinear form B, by

$$Q(z) = \frac{1}{2}B[z,z] = \left\langle A^{s}u, A^{t}v\right\rangle, \text{ for } z = (u,v) \in E \tag{2.14}$$

which yields for $z = z^{+} + z^{-}$, $z^{+} \in E^{+}$, $z^{-} \in E^{-}$, that

$$\frac{1}{2}\|z\|_{E}^{2} = Q(z^{+}) - Q(z^{-}) \tag{2.15}$$

and that, there exists a constant $c_0 > 0$, such that

$$Q(z) \geq c_0 \|z\|_{E}^{2} \text{ for } z \in E^{+} \tag{2.16}$$

and

$$Q(z) \leq -c_0 \|z\|_{E}^{2} \text{ for } z \in E^{-} \tag{2.17}$$

2.3 Rigorous variational formulation of the problem

We take $T = -\Delta_{\mathbb{H}^n}$. In order to have the Poincaré type inequality (2.8), we shall work on a bounded or a strongly asymptotically contractive domain of $\Omega \subset \mathbb{H}^n$.
Let $H = L^2(\Omega)$, $A^s = (-\Delta_{\mathbb{H}^n})^{\frac{s}{2}}$ and $E^s = D((-\Delta_{\mathbb{H}^n})^{\frac{s}{2}})$.
The appropriate functional to be associated to problem (P) in case I is

$$J(u,v) := \int_\Omega A^s u.A^t v \, d\xi - \int_\Omega [|u|^p + |v|^q] \, d\xi \tag{2.18}$$

where $s + t = 2,\ s,\ t > 0$ and

$$p \leq \frac{2N}{N-2s},\ \ q \leq \frac{2N}{N-2t}.$$

In case II we take $s = t = 1$, and hence we regain the form in (1.3).

3 Minimax theorem and Plais-Smale sequence

In this section, we establish the linking geometry of J on bounded or strongly asymptitically contractive domain of $\mathbb{H}^n$, to give a Palais-Smale sequence by the minimax principle used in [14] and [3].

Definition 3.1. Let S be a closed subset of a Banach space X, and Q a sub-manifold of X, with relative boundary ∂Q.
We say that S and ∂Q link if:

1. $S \cap \partial Q = \emptyset$.

2. $\forall h \in C^0(X,X)$ such that $h_{|\partial Q} = id$, there holds $h(Q) \cap S \neq \emptyset$.

Theorem 3.2. *Let $J : X \longrightarrow \mathbb{R}$ be a C^1 functional. Consider a closed subset $S \subset X$, and a sub-manifold $Q \subset X$, with relative boundary ∂Q. Suppose:*

1. S and ∂Q link.

2. $\exists \delta > 0$ such that

$$J(z) \geq \delta \ \forall z \in S,$$

$$J(z) \leq 0 \ \forall z \in \partial Q.$$

Let

$$\Gamma := \{h \in C^0(X,X) \mid h_{|\partial Q} = id\},$$

and

$$c := \inf_{h\in\Gamma} \sup_{z\in Q} J(h(z)) \geq \delta.$$

Then there exists a sequence $(z_k)_{k\in\mathbb{N}} \subset X$, such that

$$\begin{cases} J(z_k) \underset{k\to\infty}{\longrightarrow} c, \\ J'(z_k) \underset{k\to\infty}{\longrightarrow} 0. \end{cases} \tag{3.1}$$

We choose numbers $\mu > 1,\ \nu > 1$, such that $\dfrac{1}{p} < \dfrac{\mu}{\mu+\nu}$, and $\dfrac{1}{q} < \dfrac{\nu}{\mu+\nu}$.
The following propositions give the linking geometry of J. Their proofs are similar to those in [3] and will be omitted.

Proposition 3.3. *There exist $\rho > 0,\ \delta > 0$, such that if we define*

$$S := \{(\rho^{\mu-1}u, \rho^{\nu-1}v) \mid \|(u,v)\| = \rho,\ (u,v) \in E^+\},$$

then

$$J(z) \geq \delta \ \ \forall z \in S.$$

Proposition 3.4. *There exist $\sigma > 0,\ M > 0$, such that if we define* $Q = \{\tau(\sigma^{\mu-1}u_+, \sigma^{\nu-1}v_+) + (\sigma^{\mu-1}u, \sigma^{\nu-1}v) \mid 0 \leq \tau \leq \sigma,\ 0 \leq \|(u,v)\|_E \leq M,$ *and* $(u,v) \in E^-\}$, *where $z^+ = (u_+, v_+) \in E^+$, with u_+ some fixed eigenvector of $-\Delta_{\mathbb{H}^n}$, then*

$$J(z) \leq 0 \ \ \forall z \in \partial Q,$$

where ∂Q is the boundary of Q, relative to the subspace

$$\left\{\tau(\sigma^{\mu-1}u_+, \sigma^{\nu-1}v_+) + (\sigma^{\mu-1}u, \sigma^{\nu-1}v) \mid \tau \in \mathbb{R},\ (u,v) \in E^-\right\}.$$

Proposition 3.5. *Let $\Omega \subseteq \mathbb{H}^n$ be any bounded or unbounded domain of $\mathbb{H}^n$, and let $(z_k = (u_k, v_k))_{k\in\mathbb{N}} \subset E$ be a Palais-Smale sequence of J at level c, that is,*

$$J(z_k) \underset{k\to\infty}{\longrightarrow} c, \text{ and } J'(z_k) \underset{k\to\infty}{\longrightarrow} 0. \tag{3.2}$$

Then $(z_k)_{k\in\mathbb{N}}$ is bounded.

Proof. *Let $(z_k = (u_k, v_k))_{k\in\mathbb{N}}$ be a sequence of E satisfying (3.2). Then, there exists a sequence $\varepsilon_k \underset{k\to\infty}{\longrightarrow} 0$, such that*

$$|J'(z_k)\eta| \leq \varepsilon_k \|\eta\|_E \ \ \forall \eta \in E. \tag{3.3}$$

Taking $\eta_k = \dfrac{pq}{p+q}\left(\dfrac{1}{p}u_k, \dfrac{1}{q}v_k\right)$, and using (3.2), we obtain

$$c + \varepsilon_k \|\eta\|_E \geq J(z_k) - J'(z_k)\eta_k = \left(\frac{pq}{p+q} - 1\right)\int_\Omega |u_k|^p + |v_k|^q d\xi. \tag{3.4}$$

Hence, there exists a positive constant C such that

$$\int_{\mathbb{H}^n} |u_k|^p + |v_k|^q d\xi \leq C\left(1 + \|z_k\|_E\right) = C\left(1 + \|u_k\|_{E^s} + \|v_k\|_{E^t}\right). \tag{3.5}$$

By considering $\eta = (\phi, 0)$ *with* $\phi \in E^s$, *we obtain from (3.3)*

$$\left| \int_\Omega A^s \phi . A^t v_k d\xi \right| \leq p \int_\Omega |u_k|^{p-1} |\phi| d\xi + \varepsilon_k \|\phi\|_{E^s} \tag{3.6}$$

$$\leq C \left(\|u_k\|^{p-1}_{L^p(\Omega)} + 1 \right) \|\phi\|_{E^s}. \tag{3.7}$$

Taking $\phi = v_k$, *we obtain*

$$\|v_k\|_{E^t} \leq C \left(\|u_k\|^{p-1}_{L^p(\Omega)} + 1 \right). \tag{3.8}$$

Similar reasoning yields that

$$\|u_k\|_{E^s} \leq C \left(\|v_k\|^{q-1}_{L^q(\Omega)} + 1 \right). \tag{3.9}$$

Replacing (3.8) *and* (3.9) *into* (3.5), *we obtain*

$$\|u_k\|_{E^s} + \|v_k\|_{E^t} \leq C \left(\|u_k\|^{\frac{p-1}{p}}_{E^s} + \|v_k\|^{\frac{q-1}{q}}_{E^t} + 1 \right). \tag{3.10}$$

Since the exponents in the right-hand side of (3.10) *are less then 1, the sequence* z_k *is bounded in* E.

□

4 Existence in the subcritical case on a strongly asymptotically contractive domain

Proof of Theorem 1.2.
By Theorem 3.2, we obtain a critical sequence (z_k) on Ω satisfying equation (3.1). According to Proposition 3.5, the sequence is bounded, therefore a relabeled subsequence converges weakly to a limit z. Sinc J' is continuous in the weak topology $J'(z) = 0$.
We claim that $z \neq 0$. Suppose it were, then by compact Sobolev embeddings,

$$\int_{\Omega} [|u_k|^p + |v_k|^q] \, d\xi \to 0. \tag{4.1}$$

We also have $J'(z_k)z_k \to 0$ since z_k is bounded, but combining this with equation (4.1), one obtains that $\|z_k\|^2 \to 0$ contradicting (3.1) since $c > 0$.

□

5 Existence in the critical case on $\mathbb{H}^n$

5.1 Abstract concentration compactness

In this section, we recall the abstract concentration compactness due to I.Schindler, and K.Tintarev [16], and we give a version adapted to our problem.
Let H be a separable Hilbert space, and let D be a bounded multiplicative group of bounded linear operators on H.

Definition 5.1. We say that D is a set of dislocations if it satisfies the following conditions:

P1) Let $g_k \in D$. If $g_k \not\rightharpoonup 0$, and if $u_k \rightharpoonup 0$, then there exists a subsequence such that $g_k u_k \rightharpoonup 0$.

P2) If there exists a $u \in H \setminus \{0\}$ such that $g_k u \rightharpoonup 0$, then $g_k \rightharpoonup 0$.

P3) If $g_k \in D$, and $u_k \rightharpoonup 0$, then $g_k^* g_k u_k \rightharpoonup 0$.

Definition 5.2. Let $u,\ u_k \in H$. We say that u_k converges to u weakly with concentration, and we note $u_k \overset{D}{\rightharpoonup} u$, if $\forall \phi \in H^*$

$$\lim_{k\to\infty} \sup_{g \in D} (g(u_k - u), \phi) = 0.$$

If D is a compact group, concentrated weak convergence is equivalent to weak convergence.

Theorem 5.3. *Let $(u_k)_{k\in\mathbb{N}} \subset H$ be a bounded sequence, and let D be a set of dislocations. Then there exist $(w^{(n)})_{n\in\mathbb{N}} \subset H$, $(g_k^{(n)})_{n\in\mathbb{N}} \subset D$, $k \in \mathbb{N}$, such that for a renamed subsequence,*

$$g_k^{(1)} = id, \quad g_k^{(n)^{-1}} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m,$$

$$w^{(n)} = w - \lim_k g_k^{(n)^{-1}} u_k,$$

$$u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0.$$

5.2 Concretisation of the abstract concentration compactness on E

Let D be the infinite multiplicative group of bounded linear operators defined on E by

$$\begin{aligned} g_{\lambda,\alpha}(u(\xi), v(\xi)) &= \left(\lambda^{\frac{2n+2}{p}} u(\alpha \circ \delta_\lambda \xi), \lambda^{\frac{2n+2}{q}} v(\alpha \circ \delta_\lambda \xi)\right) \\ &= \left(g^1_{\lambda,\alpha} u, g^2_{\lambda,\alpha} v\right) \end{aligned}$$

where $\alpha \in \mathbb{R}^{2n+1}$, and λ is a positive real.

Lemma 5.4. *D is a set of dislocations.*

Proof. *Let $z := (u, v) \in \mathcal{H}$. Properties* **P1), P2)** *are clearly satisfied since we observe that*

$$g_k := g_{\lambda_k,\alpha_k} \rightharpoonup 0 \Longleftrightarrow \alpha_k \longrightarrow \infty, \text{ or } \lambda_k \longrightarrow 0, \text{ or } \lambda_k \longrightarrow \infty.$$

Observe that $g_k^* = g_k^{-1}$, which yields **P3)**.

□

The following theorem is a corollary of Theorem 5.3. See [15].

Theorem 5.5. *Let $(z_k = (u_k, v_k))_k$ be a bounded sequence in E. Then for a renamed subsequence, there exist $w^{(1)}, w^{(2)}, \dots \in E$, and $(\alpha_k^{(1)}, \lambda_k^{(1)}), (\alpha_k^{(2)}, \lambda_k^{(2)})$, $\dots \in \mathbb{H}^n \times \mathbb{R}_*^+$, such that*

$$w^{(n)} = w - \lim_{k\to\infty} g_{\frac{1}{\lambda_k^{(n)}}, -\alpha_k^{(n)}} z_k,$$

and for $r \neq m$

$$\lambda_k^{(r)}/\lambda_k^{(m)} \to \infty, \text{ or } \lambda_k^{(r)}/\lambda_k^{(m)} \to 0, \quad \text{or } |\alpha_k^{(r)} - \alpha_k^{(m)}| \to \infty.$$

The series $\sum_n g_{\lambda_k^{(n)},\alpha_k^{(n)}} w^{(n)}$ *converges absolutely in* E*, and:*

$$z_k - \sum_n g_{\lambda_k^{(n)},\alpha_k^{(n)}} w^{(n)} \overset{D}{\rightharpoonup} 0.$$

Lemma 5.6. *Let* $(u_k)_k$ *be a bounded sequence in* $S_1^2(\mathbb{H}^n) \cap L^p(\mathbb{H}^n)$*.*
If $u_k \overset{D}{\rightharpoonup} 0$*, then modulo a subsequence,* $\lim_{k\to\infty} \|u_k\|_{L^p(\mathbb{H}^n)} = 0$*, for all* $p \geq 1$*.*

For the proof we need the following Sobolev embedding theorem [18]:

Theorem 5.7. *Let* Ω *be a bounded open set of* $\mathbb{H}^n$*. Then* $\overset{\circ}{S_1^2}(\Omega)$ *is compactly embedded in* L^p*, for* $1 \leq p < N^*$*.*

We need, also the following lemma, which is an adapted version of Lemma 1.1 in [10]:

Lemma 5.8. *Let* $(u_k)_{k\in\mathbb{N}}$ *be a bounded sequence in* $S_1^2(\mathbb{H}^n)$*, such that, for some* $R > 0$

$$\liminf_{k\to\infty} \sup_{y\in\mathbb{H}^n} \int_{B(y,R)} u_k^2 d\xi = 0 \tag{5.1}$$

Then $u_k \xrightarrow[k\to\infty]{} 0$ *in* $L^s(\mathbb{H}^n)$*, for* $2 < s < N^*$*.*

Proof of Lemma 5.6. Note that $u_k \overset{D}{\rightharpoonup} 0 \Longrightarrow \forall g \in D : gu_k \rightharpoonup 0$.

We will give the proof in two steps, the first step deal with the case $1 \leq p < N^*$, and the second deal with the case $p \geq N^*$:

Let $1 \leq p < N^*$, $\alpha \in \mathbb{H}^n$, $g = g_{1,-\alpha}$, and $B = B(\xi, R)$ be a ball of center ξ and radius R.
According to the Theorem 5.7, we obtain:

$$\|g^1_{1,-\alpha} u_k\|^p_{L^p(B)} \leq C \|g^1_{1,-\alpha} u_k\|^2_{\overset{\circ}{S_1^2}(B)} \|g^1_{1,-\alpha} u_k\|^{p-2}_{L^p(B)} \tag{5.2}$$

Let $\{B = B(\xi, R),\ \xi \in Z\}$ be a finite cover for $\mathbb{H}^n$. So, by summing inequalities (5.2) over $\xi \in Z$, we obtain:

$$\|u_k\|^p_{L^p(\mathbb{H}^n)} \leq C \|u_k\|^2_{S_1^2(\mathbb{H}^n)} \sup_{\xi\in Z} \|g^1_{1,-\alpha} u_k\|^{p-2}_{L^p(B(\xi,R))}$$

By the compactness of the embedding of $\overset{\circ}{S_1^2}(B)$ into $L^p(B)$, it follows that, modulo a subsequence, $g^1_{1,-\alpha} u_k \xrightarrow[k\to\infty]{} 0$ in $L^p(B)$.
Hence, $\|u_k\|_{L^p(\mathbb{H}^n)} \xrightarrow[k\to\infty]{} 0$ for $1 \leq p < N^*$.

Let us now deal with the case $p \geq N^*$.

Let $g = g_{\lambda_k^{-1},0}$, where $\lambda_k \xrightarrow[k\to+\infty]{} +\infty$ is chosen such that

$$\int_{|u_k|>\lambda_k^{\frac{2n+2}{p}}} |u_k|^p \longrightarrow 0;$$

and let $w_k(\xi) = g^2_{\lambda_k^{-1},0} u_k(\xi) = \lambda_k^{-\frac{2n+2}{p}} u_k(\delta_{\lambda_k^{-1}}\xi)$, i.e $u_k(\xi) = \lambda_k^{\frac{2n+2}{p}} w_k(\delta_{\lambda_k}\xi)$.
We have

$$\int_{|u_k|<\lambda_k^{\frac{2n+2}{p}}} |u_k|^p d\xi = \int_{|w_k|<1} |w_k|^p d\xi \le \int_{\mathbb{H}^n} |w_k(x)|^s d\xi, \text{ where } 2 < s < N^* \le p.$$

Note that the hypothesis (5.1) is satisfied if we take $g = g_{1,y}$. So, according to Lemma 5.8, $\int_{\mathbb{H}^n} |w_k(x)|^s d\xi \xrightarrow[k\to\infty]{} 0$.
Hence $\|u_k\|_{L^p(\mathbb{H}^n)} \xrightarrow[k\to\infty]{} 0$ for $p \ge N^*$.

□

5.3 Proof of the main result

Proof of Theorem 1.3. To be able to use the linking geometry stated in Proposition 3.3 and Proposition 3.4, we shall work on a bounded domain Ω of $\mathbb{H}^n$. Note that $E^1 = \overset{\circ}{S}{}^2_1$.
By Theorem 3.2, we obtain a critical sequence (z_k) on Ω. Sinc $\overset{\circ}{S}{}^2_1 (\Omega) \subset \overset{\circ}{S}{}^2_1 (\mathbb{H}^n)$ we may consider $(z_k) \subset \overset{\circ}{S}{}^2_1 (\mathbb{H}^n)$. Since J is invariant under the action of $g_{\lambda,\alpha}$, we conclude that the sequence $(g_{\lambda,\alpha} z_k)$, that we denote again by (z_k), is a critical sequence on $\mathbb{H}^n$, and according to Proposition 3.5, it is bounded.
By Theorem 5.5, there exist $w^{(1)}, w^{(2)}, \cdots \in E$, and $(\alpha_k^{(1)}, \lambda_k^{(1)})$, $(\alpha_k^{(2)}, \lambda_k^{(2)}) \cdots \in \mathbb{H}^n \times \mathbb{R}^+_*$, such that

$$z_k - \sum_n g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0,$$

where $g_k^{(n)} = g_{\lambda_k^{(n)},\alpha_k^{(n)}}$.
(z_k) does not converge weakly with concentration to 0. In fact, if we suppose that $z_k \overset{D}{\rightharpoonup} 0$, we will have by Lemma 5.6, $\lim_{k\to\infty} \|z_k\|_{L^p(\mathbb{H}^n)\times L^q(\mathbb{H}^n)} = 0$ (modulo a subsequence), which shows that $J(z_k) \longrightarrow 0$. Contradiction. Then there exists a $w^{(n_0)} \neq 0$.
On the other hand, for some $g_k \in D$, we have

$$g_k z_k \rightharpoonup w^{(n_0)}.$$

Then,

$$J'(g_k z_k) \rightharpoonup J'(w^{(n_0)})$$

However, $J'(z_k) \xrightarrow[k\to\infty]{} 0 \Longrightarrow J'(g_k z_k) \xrightarrow[k\to\infty]{} 0$. Then, $J'(w^{(n_0)}) = 0$.

□

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