American. Jr. of Mathematics and Sciences
Vol. 1, No.1 ,(January 2012)

www.journalshub.com

MAX-MIN AND MIN-MAX VALUES OF VARIOUS MEASURES OF FUZZY DIVERGENCE

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ABSTRACT

In the present manuscript, we have considered some new non-probabilistic (fuzzy) measures of directed divergence, and keeping in view the importance and areas of applications of these measures, we have investigated their optimum values.

Keywords: Distance measure, Fuzzy entropy, Fuzzy directed divergence, Convexity.

INTRODUCTION

The measure of distance is an important term that describes the difference between fuzzy sets and can be considered as a dual concept of similarity measure. Many researchers have used distance measure to define fuzzy entropy. Using the axiom definition of distance measure, Fan, Ma and Xie [2] developed some new formulas of fuzzy entropy induced by distance measure and studied some new properties of distance measure. Rosenfeld [10] defined the shortest distance between two fuzzy sets as a density function on the non - negative reals. Corresponding to the probabilistic measure of divergence due to Kullback and Leibler [5], Bhandari and Pal [1] introduced the following measure of fuzzy directed divergence:

$$I(A:B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right]$$

(1.1)

Corresponding to Renyi's [9] and Havrada and Charvat's [3] divergence measures, Kapur [4] took the following expressions of measures of fuzzy directed divergence:

$$D^{\alpha}(A:B)=\frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left[\mu_A^{\alpha}(x_i)\,\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\right] \quad (1.2)$$

$$D_{\alpha}(A:B)=\frac{1}{\alpha-1}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\,\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\,(1-\mu_B(x_i))^{1-\alpha}-1\right]$$

(1.3)

Tran and Duckstein [11] developed a new approach for ranking fuzzy numbers based on a distance measure. Parkash [6] introduced a generalized fuzzy divergence, given by

$$D_{\alpha}^{\beta}(A,B)=[(\alpha-1)\,\beta]^{-1}\sum_{i=1}^{n}\left[\left\{\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\right\}^{\beta}-1\right]$$

(1.4)

Many measures of fuzzy divergence along with their detailed properties and important applications have been discussed by various authors including those of Kapur [4], Parkash [6], Parkash and Sharma [7], Parkash and Tuli [8] etc. In fact, Kapur [4] has developed many expressions for the measures of fuzzy directed divergence corresponding to probabilistic measures of divergence due to Harvada and Charvat [3], Renyi [9] etc.

2. OPTIMIZATION OF VARIOUS MEASURES OF DIVERGENCE

In this section, we consider Renyi's [9] measure of fuzzy directed divergence given in (1.2) and examine it for its maximum and minimum values.

I. Minimum values of Renyi's [9] measure of fuzzy directed divergence

We now find the minimum value of $D^{\alpha}(A:B)$. Since $D^{\alpha}(A:B)$ is a convex function, its minimum value exists. For minimum value, we put

$$\frac{\partial D^{\alpha}(A:B)}{\partial \mu_A(x_i)} = 0 \text{ which gives } \mu_A(x_i) = 1-\mu_A(x_i) \Rightarrow \mu_A(x_i) = \frac{1}{2}$$

Now $\sum_{i=1}^{n} \mu_A(x_i) = k$ gives $\frac{n}{2} = k \Rightarrow \frac{k}{n} = \frac{1}{2} = \mu_A(x_i) \forall i$

$$\therefore \text{Min.}D^{\alpha}(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left[\left(\frac{k}{n}\right)^{\alpha}\mu_B^{1-\alpha}(x_i) + \left(1-\frac{k}{n}\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]$$

Case-I. When $k=0$, then

$$\text{Min.}D^{\alpha}(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left(1-\mu_B(x_i)\right)^{1-\alpha} = -\sum_{i=1}^{n}\log\left(1-\mu_B(x_i)\right)$$

Case-II. When $k=\frac{n}{2}$, then

$$\text{Min.}D^{\alpha}(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left[\frac{\left(\mu_B(x_i)\right)^{1-\alpha} + \left(1-\mu_B(x_i)\right)^{1-\alpha}}{2^{\alpha}}\right]$$

Case-III. When $k=n$, then

$$\text{Min.}D^{\alpha}(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left(\mu_B(x_i)\right)^{1-\alpha} = -\sum_{i=1}^{n}\log\left(\mu_B(x_i)\right)$$

Illustration. To illustrate the above process, we consider $\mu_B(x_i) = \left(\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n+1}\right)$

For $k=0$,

$$\text{Min.}D^{\alpha}(A:B) = -\sum_{i=1}^{n}\log\left(1-\mu_B(x_i)\right) = -\left(\log\frac{1}{2}+\log\frac{2}{3}+\log\frac{3}{4}+\ldots.\log\frac{n}{n+1}\right)$$

$$= \left(\log 2+\log\frac{3}{2}+\log\frac{4}{3}+\ldots.\log\frac{n+1}{n}\right)$$

$$= \left(\log 2.\frac{3}{2}.\frac{4}{3}.\ldots..\frac{n+1}{n}\right) = \log(n+1)$$

For $k=\frac{n}{2}$,

$$\text{Min.}D^{\alpha}(A:B) = \frac{1}{\alpha-1}\left[\log\frac{2^{\alpha-1}+2^{\alpha-1}}{2^{\alpha}} + \log\frac{3^{\alpha-1}+\left(\frac{3}{2}\right)^{\alpha-1}}{2^{\alpha}} + \log\frac{4^{\alpha-1}+\left(\frac{4}{3}\right)^{\alpha-1}}{2^{\alpha}} + \log\frac{(n+1)^{\alpha-1}+\left(\frac{n+1}{n}\right)^{\alpha-1}}{2^{\alpha}}\right]$$

$$= \log\frac{(n+1)!}{2^{n\alpha}} + \frac{1}{\alpha-1}\left[\log\left(1+\frac{1}{1^{\alpha-1}}\right).\left(1+\frac{1}{2^{\alpha-1}}\right).\left(1+\frac{1}{3^{\alpha-1}}\right).\ldots..\left(1+\frac{1}{n^{\alpha-1}}\right)\right]$$

For $k=n$,

$$\text{Min.}D^{\alpha}(A:B)=\frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left(\mu_B(x_i)\right)^{1-\alpha}=-\sum_{i=1}^{n}\log\left(\mu_B(x_i)\right)$$

$$=\log 2.3.4.......(n+1)=\log(n+1)!$$

II. Maximum values of Renyi's [9] measure of fuzzy directed divergence

We now find maximum value of $D^{\alpha}(A:B)$.

Case-I: When k is any +ve integer, then we can choose k values of $\mu_A(x_i)$ as unity and others $(n-k)$ as 0, that is, $\mu_A(x_i)=\{1,1,1,....,1,0,0,....,0\}$. Now, we can write

$$D^{\alpha}(A:B)=\frac{1}{\alpha-1}\left[\sum_{i=1}^{m}\log\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]+\sum_{i=m+1}^{n}\log\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]\right]$$

Thus, the maximum value of $D^{\alpha}(A:B)$ is given as

$$\text{Max.}D^{\alpha}(A:B)=\frac{1}{\alpha-1}\left[\sum_{i=1}^{m}\log\mu_B^{1-\alpha}(x_i)+\sum_{i=m+1}^{n}\log\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]$$

Illustration. To illustrate the above process, we consider $\mu_B(x_i)=\left(\frac{1}{2},\frac{1}{3},\frac{1}{4},.......\frac{1}{n+1}\right)$

For $m=0$, $\text{Min.}D^{\alpha}(A:B)=\frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left(1-\mu_B(x_i)\right)^{1-\alpha}=-\sum_{i=1}^{n}\log\left(1-\mu_B(x_i)\right)$

$$=\left(\log 2.\frac{3}{2}.\frac{4}{3}.....\frac{n+1}{n}\right)=\log(n+1)$$

For $m=1$**,**

$$\text{Min.}D^{\alpha}(A:B)=\frac{1}{\alpha-1}\left[\log\left(\mu_B^{1-\alpha}(x_1)\right)+\sum_{i=2}^{n}\log\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]$$

$$=\left(\log 2+\log\frac{3}{2}+\log\frac{4}{3}+....\log\frac{n+1}{n}\right)$$

$$=\left(\log 2.\frac{3}{2}.\frac{4}{3}.....\frac{n+1}{n}\right)=\log(n+1)$$

For $m=2$,

$$\text{Max.}D^{\alpha}(A:B)=\log 2+\log 3+\log\frac{4}{3}+\log\frac{5}{4}+....\log\left(\frac{n+1}{n}\right)=\log 2!+\log(n+1)$$

For $m=3$,

$$\text{Max.}D^{\alpha}(A:B)==\frac{1}{\alpha-1}\left(\log 2^{\alpha-1}+\log 3^{\alpha-1}+\log 4^{\alpha-1}+\log\left(\frac{5}{4}\right)^{\alpha-1}+\log\left(\frac{6}{5}\right)^{\alpha-1}+....\log\left(\frac{n+1}{n}\right)^{\alpha-1}\right)$$

$$=\log 2+\log 3+\log 4+\log\frac{5}{4}+\log\frac{6}{5}....\log\left(\frac{n+1}{n}\right)=\log 3!+\log(n+1)$$

For $m=n$,

$$\text{Max.}D^{\alpha}(A:B)=\frac{1}{\alpha-1}\sum_{i=1}^{n}\log\left(\mu_B(x_i)\right)^{1-\alpha}=-\sum_{i=1}^{n}\log\left(\mu_B(x_i)\right)$$

$$=-\left(\log\frac{1}{2}+\log\frac{1}{3}+\log\frac{1}{4}+....\log\frac{1}{n+1}\right)=\left(\log 2.3.4....(n+1)\right)$$

$$=\log(n+1)!$$

The above values show that $\text{Max.}D^{\alpha}(A:B)$ is piecewise convex function.

Case-II: If k is any fraction, then, we can write $k=m+\xi$, where m is a +ve integer and ξ is a positive fraction. We can choose m fuzzy values of $\mu_A(x_i)$ as unity, $(m+1)^{th}$ value of $\mu_A(x_i)$ as ξ and remaining $(n-m-1)$ values of $\mu_A(x_i)$ as 0, that is, $\mu_A(x_i)=\{1,1,...,1,\xi,0,0,....,0\}$.Thus,

$$D^{\alpha}(A:B)=\frac{1}{\alpha-1}\left[\sum_{i=1}^{n}\log\mu_B^{1-\alpha}(x_i)+\log\xi^{\alpha}\mu_B^{1-\alpha}(x_{m+1})+(1-\xi)^{\alpha}\left(1-\mu_B(x_{m+1})\right)^{1-\alpha}+\sum_{i=m+2}^{n}\log\left(1-\mu_B(x_i)\right)^{1-\alpha}\right]$$

$$=-\sum_{i=1}^{m}\log\mu_B(x_i)-\sum_{i=m+2}^{n}\log\left(1-\mu_B(x_i)\right)+\frac{1}{\alpha-1}\log\left[\xi^{\alpha}\mu_B^{1-\alpha}(x_{m+1})+(1-\xi)^{\alpha}\left(1-\mu_B(x_{m+1})\right)^{1-\alpha}\right]$$

To illustrate the above process, we consider $\mu_B(x_i)=\left(\frac{1}{2},\frac{1}{3},\frac{1}{4},.......\frac{1}{n+1}\right)$

$$\therefore \text{Max.}D^{\alpha}(A:B)=\log 2+\log 3+........\log(m+1)+\log\frac{m+3}{m+2}+\log\frac{m+4}{m+3}+.........\log\frac{n+1}{n}$$

$$+\frac{1}{\alpha-1}\log\left[\xi^{\alpha}(m+2)^{\alpha-1}+(1-\xi)^{\alpha}\left(\frac{m+2}{m+1}\right)^{\alpha-1}\right]$$

$$=\log(m+1)!+\log\frac{n+1}{m+1}+\frac{1}{\alpha-1}\log\left[\xi^{\alpha}(m+1)^{\alpha-1}+(1-\xi)^{\alpha}\right]$$

$$=\log(n+1)m!+\phi(\xi)$$

where $\phi(\xi)=\frac{1}{\alpha-1}\log\left[\xi^{\alpha}(m+1)^{\alpha-1}+(1-\xi)^{\alpha}\right]$

We, now check the convexity of $\phi(\xi)$.

Take $h(\xi)=\xi^{\alpha}(m+1)^{\alpha-1}+(1-\xi)^{\alpha}$,

Thus $\frac{d}{d\xi}h(\xi)=\alpha\left[\xi^{\alpha-1}(m+1)^{\alpha-1}-(1-\xi)^{\alpha-1}\right]$

Also $\frac{d^2}{d\xi^2}h(\xi)=\alpha(\alpha-1)\left[\xi^{\alpha-2}(m+1)^{\alpha-1}+(1-\xi)^{\alpha-2}\right]$

$$=\left(\alpha^2-\alpha\right)\left[\xi^{\alpha-2}(m+1)^{\alpha-1}+(1-\xi)^{\alpha-2}\right]$$

If $\alpha>1$, then $\frac{d^2}{d\xi^2}h(\xi)>0\Rightarrow h(\xi)$ is a convex function of ξ

$\Rightarrow \log h(\xi)$ is a convex function of ξ

$\Rightarrow \frac{1}{\alpha-1}\log h(\xi)$ is a convex function of ξ for $\alpha>1$, If $\alpha < 1$, then $\frac{d^2}{d\xi^2} h(\xi) < 0 \Rightarrow h(\xi)$ is a concave function of ξ

$\Rightarrow \log h(\xi)$ is a concave function of ξ

$\Rightarrow \frac{1}{\alpha - 1} \log h(\xi)$ is a convex function of ξ for $0 < \alpha < 1$,

Hence, $\phi(\xi)$ is a convex function of ξ for each α. Its minimum value exists at

$$\xi = \frac{1}{m+2}$$

Also, the minimum value of $\mathrm{Max.D}^{\alpha}(A:B)$ is given as

$$\mathrm{Min.Max.D}^{\alpha}(A:B) = \log(n+1)m! + \phi\left(\frac{1}{m+2}\right)$$

$$= \log(n+1)m! + \frac{1}{\alpha - 1} \log\left(\frac{(m+1)^{\alpha-1}}{(m+2)^{\alpha}} + \frac{(m+1)^{\alpha}}{(m+2)^{\alpha}}\right)$$

$$= \log(n+1)m! + \log\frac{(m+1)}{(m+2)}$$

$$= \log(n+1) + \log\frac{(m+1)m!}{(m+2)} = \log(n+1) + \log\frac{(m+1)!}{(m+2)}$$

For $m = 0$, $\mathrm{Min.Max.D}^{\alpha}(A:B) = \log(n+1) + \log\frac{1!}{2}$

For $m = 1$, $\mathrm{Min.Max.D}^{\alpha}(A:B) = \log(n+1) + \log\frac{2!}{3}$

For $m = 2$, $\mathrm{Min.Max.D}^{\alpha}(A:B) = \log(n+1) + \log\frac{3!}{4}$

For $m = 3$, $\mathrm{Min.Max.D}^{\alpha}(A:B) = \log(n+1) + \log\frac{4!}{5}$

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For $m = n$, For $m = 2$, $\mathrm{Min.Max.D}^{\alpha}(A:B) = \log(n+1) + \log\frac{(n+1)!}{n+2}$

Conclusions: When $\mu_B(x_i)$ is a monotonically decreasing, the Max. cross entropy is an increasing function. It can also be proved that if $\mu_B(x_i)$ is a monotonically increasing, the Max. cross entropy is a decreasing function. Also in both cases, $\mathrm{Max.D}^{\alpha}(A:B)$ is a piecewise convex function of k. In the literature of distance measures, there exist many parametric and non-parametric measures of fuzzy divergence introduced by various researchers. Proceeding on similar way as done in section 2, the optimum values of other divergence measures can be studied.

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