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Cosets and Ideals of a Regular Incline

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Abstract

In this paper, we have proved that the set of all ideals in a regular incline, R forms an incline under the operations I+J and IJ, where I and J are ideals as a subincline of R. We have proved that the coset of a regular element in an incline with respect to an ideal I is a subset of the set of all 1-inverses of that element. Further we have proved that if two cosets are equal then the corresponding elements are related with respect to I.

Keywords: incline, subincline, ideals, cosets.

Mathematics Subject Classification: 16D25, 16Y60.

1. Introduction

Inclines are additively idempotent semirings in which products are less than (or) equal to either factor. The concept of incline was introduced by Cao and later it was developed by Cao et.al. in [2]. Recently a survey on incline was made by Kim and Roush in [3]. Incline algebra is a generalization of both Boolean and fuzzy algebras and it is a special type of semiring. It has both Semiring structure and Poset structure.

In [2], Cao, Kim and Roush have proved that "If two sided ideals in a semiring with 0 commute then they form an incline under the operations M+N and MN, where M and N are ideals of the semiring" (Theorem 6.4.1 of [2]). Here we have proved that this result is valid only for a regular incline R that is, whose elements are all idempotents. We have illustrated with a suitable example. In section 2, we present the basic definitions and required results of an incline R. In section 3, we have proved that the set of all ideals in a regular incline R, forms an incline under the operations I+J and IJ, where I and J are ideals as a subincline of R. In section 4, we have introduced the concept of cosets with respect to an ideal of an incline as an extension of the cosets in a ring [6]. We have proved that C(a), the coset of a regular element 'a' in an incline is a subset of a{1}, the set of all 1-inverses of 'a'. It is well known that "The set of all cosets in a ring with respect to an ideal will form a partition of the ring" (p.187 Theorem B of [6]). Here we have illustrated with an example that this fails for an incline. We have proved that if two cosets are equal then the corresponding elements are related with respect to I. We have exhibited that coset of an element 'x' in an incline is a subset of $[x]_I$, the equivalence class of 'x' with respect to I.

2. Preliminaries

In this section we present some definitions and required results in an incline.

Definition 2.1

An *incline* is a non-empty set R with binary operations addition and multiplication denoted as +, · defined on R x R → R such that for all x, y, z ∈ R,

$x+y = y+x$, $x+(y+z) = (x+y)+z$,
$x(y+z) = xy+xz$, $(y+z)x = yx+zx$,
$x(yz) = (xy)z$, $x+xy = x$,
$x+x = x$, $y+xy = y$.

An incline R is said to be *commutative* if xy = yx, for all x, y ∈R.

In [2], Authors have studied on commutative inclines and in [5] authors deal with non-commutative inclines.

Definition 2.2

An element 0_R in an incline R is the *zero element* of R if $x + 0_R = 0_R + x = x$ and $x \,.\, 0_R = 0_R \,.\, x = 0_R$, for all x ∈ R.

Definition 2.3

$(R, \le)$ is an incline with order relation '$\le$' defined as, $x \le y$ if and only if $x+y = y$, for $x, y \in R$. If $x \le y$ then y is said to *dominate* x.

Property 2.4

For x, y in an incline R, $x+y \ge x$ and $x+y \ge y$.

Property 2.5

For x, y in an incline R, $xy \le x$ and $xy \le y$.

Remark 2.6

In an incline R, if R has the zero element 0_R then by Property 2.4 and Definition 2.2, it follows that $0_R \le x$ for all $x \in R$. Thus 0_R is the least element of R.

Definition 2.7

An element a in an incline R is said to be regular, if there exists an element x in R such that a = axa. An incline R is said to be regular, if and only if every element in R is regular.

Lemma 2.8 [5]

An incline R is *regular* if and only if $x^2 = x$, for all $x \in R$.

Proposition 2.9 [5]

If a is regular, then a is the smallest g-inverse of a, that is $a \le x$, for all $x \in a\{1\} = \{x / axa = a\}$.

Definition 2.10

A *subincline* of an incline R is a non–empty subset I of R which is closed under the incline operations addition and multiplication.

Definition 2.11

A subincline I is said to be an *ideal* of an incline R if $x \in I$ and $y \le x$ then $y \in I$.
We call it as "ideal as a subincline".

Definition 2.12 [1]

For any $x, y \in R$, a relation '$\sim$' on R is defined by $x \sim y$ with respect to an ideal I of R if and only if there exist $i_1, i_2 \in I$ such that $x + i_1 = y + i_2$.
In [1], it is proved that the relation '$\sim$' is an equivalence relation on R.

Remark 2.13

In an incline R, for $x, y \in R$, $x \sim y$ if and only if $[x]_I = [y]_I$ and $x \nsim y$ if and only if $[x]_I$ and $[y]_I$ are disjoint, where $[x]_I = \{ y \in R / x \sim y$, with respect to $I\}$ is the equivalence class of x, with respect to an ideal I of R .

3. Ideals in a regular incline

In this section, we have proved that the set of all ideals in a regular incline R, forms an incline under the operations I+J and IJ, where I and J are ideals as a subincline of R.

Theorem 3.1

The set of all ideals in a regular incline R, forms an incline under the operations (i) $I+J = \{x+y / x \in I$ and $y \in J\}$
(ii) $IJ = \{xy / x \in I$ and $y \in J\}$

Proof

Let R be a regular incline and I, J be arbitrary elements of $\vartheta(R)$, the set of all ideals of R. We have to prove that $\vartheta(R)$ is an incline.
First let us prove that the operations I+J and IJ are well defined.
To prove I + J is an ideal of R:
Let u+v, m+n $\in$ I+J, for some u, m $\in$ I and v, n $\in$ J. Since I and J are ideals of R, p = u+m $\in$ I and q = v+n $\in$ J.

$$\text{Now, } (u+v)+(m+n) = u+m+v+n = p+q \in I + J.$$

$$(u+v)(m+n) = um + un + vm + vn = r + s, \text{ where } r = um + un \in I \text{ and } s = vm + vn \in J. \in I + J.$$

Therefore, I + J is a subincline of R.
Let x be an arbitrary element of Rand $x \leq u+v$, for some u $\in$ I and v $\in$ J. We have to show that x $\in$ I + J.
Since, x $\in$R and u $\in$ I, we have xu $\in$ R.
By incline property (2.5), $xu \leq u$.
Since I is an ideal of R, by Definition (2.11), xu $\in$ I.
Similarly, we have xv $\in$ J.
Therefore, xu + xv $\in$ I + J, implies x(u+v) $\in$ I + J →(3.1)

$$\text{Now, } x \leq u+v \Rightarrow x.x \leq x(u+v) \Rightarrow x \leq x(u+v) \Rightarrow x + x(u+v) = x(u+v)$$

Hence, by incline axiom, x + xu + xv = x and by (3.1), x = x(u+v) $\in$ I + J.
Thus, I + J is an ideal of R.
To prove IJ is an ideal of R:
Let uv, mn $\in$ IJ, for some u, m $\in$ I and v, n $\in$ J. Since I and J are ideals of R um $\in$ I and vn $\in$ J.
By Lemma 2.8, we have, uv + mn = (uv +mn)(uv +mn),
$\in$ IJ
and (uv)(mn) $\in$ IJ.
Therfore, IJ is a subincline of R.
Let x be an arbitrary element of R and $x \leq uv$, for some u $\in$ I and v $\in$ J.
We have, $x \leq uv \leq u \in I \Rightarrow x \in I$
Similarly, $x \leq uv \leq v \in J \Rightarrow x \in J$
Hence xx $\in$ IJ. By Lemma 2.8, it follows that x $\in$ IJ.
Therefore, IJ is an ideal of R.
Thus the operations on $\vartheta(R)$ are well defined.
Now let I, J, K be the arbitrary elements of $\vartheta(R)$.

1) $I+J = \{ x+y \,/\, x \in I \text{ and } y \in J\} = \{ y+x \,/\, y \in J \text{ and } x \in I\} = J+I.$
2) $I+(J+K) = \{ x+(y+z) \,/\, x \in I, y \in J \text{ and } z \in K\} = \{ (x+y)+z \,/\, x \in I, y \in J \text{ and } z \in K\} = (I+J)+K.$
3) $I(JK) = \{ x(yz) \,/\, x \in I, y \in J \text{ and } z \in K\} = \{ (xy)z \,/\, x \in I, y \in J \text{ and } z \in K\} = (IJ)K.$
4) $I(J+K) = \{ x(y+z) \,/\, x \in I, y \in J \text{ and } z \in K\} = \{ xy + xz \,/\, x \in I, y \in J \text{ and } z \in K\} = IJ + IK.$
5) $(J+K)I = \{ (y+z)x \,/\, x \in I, y \in J \text{ and } z \in K\} = \{ yx + zx \,/\, x \in I, y \in J \text{ and } z \in K\} = JI + KI.$
6) $I+I = \{ x+y \,/\, x, y \in I\}$

= { m / m = x+y ∈ I}
= I

7) I+IJ = { x+my / x, m∈I and y∈J}
By incline property (2.5), we have my ≤ m and by Definition (2.11), my ∈ I.
Therefore, x+my ∈ I.
Thus, I+IJ = I.

8) Similarly, we have J+IJ = J.

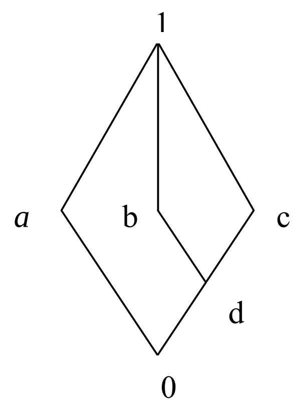
Thus, ϑ(R) is an incline. □

Remark 3.2

In the above Theorem 3.1, incline R to be regular is essential. This is illustrated in the following example.

Example 3.3

Let R={0,a,b,c,d,1} be an incline. Define •: R • R → R by x • y = d for all x,y∈{b,c,d,1} and 0 otherwise. Since 0 and d are the only idempotent elements of R, by Lemma 2.8, (R, +, •) is a commutative, non-regular incline.



Let I = {0, b, d} and J = {0, c, d} be two ideals of R.
Here, I + J = {0, b, d} + {0, c, d} = {0, b, c, d, 1}.
For the element 1∈ I + J and a ≤ 1, a ∉ I + J.
Hence, by Definition (2.11) I + J is not an ideal of R.

Remark 3.4

In Example (3.3), I, J ∈ ϑ(R) are two sided ideals in the incline R, a special type of a semiring but I + J ∉ ϑ(R). Therefore, ϑ(R) is not an incline and Theorem 6.4.1 of [2] fails.

4. Cosets of an Incline

In this section, we have introduced the concept of cosets with respect to an ideal of an incline as an extension of the cosets in a Ring [6]. We have proved that C(a), the coset of a regular element 'a' in an incline is a subset of a{1}, the set of all 1-inverses of 'a'. Further we have studied the relationship between cosets and the equivalence classes of elements with respect to I.

Definition 4.1

For a ∈ R, an incline and I is an ideal of R, a+I = {a+x / x∈I} and I+a = {x+a / x∈I}. Since an incline is additively commutative C(a) = a+I = I+a is called the coset of 'a' with respect to I.
In particular, for a∈I we get C(a) ⊆ I.

Proposition 4.2

In an incline R, for a∈R, 'a' is the least element of C(a).

Proof

Let ‘a’ be an arbitrary element of an incline R.
Now, $y \in C(a)$ implies $y = a+x$, for some $x \in I$. Then by incline Property 2.4, we have $a \leq y$, for all $y \in C(a)$. Therefore, ‘a’ is the least element of C(a). □

Lemma 4.3
In an incline for $a \in R$, if I is an ideal of R then $C(a) \subseteq I \Leftrightarrow a \in I$.

Proof
From Definition 4.1, if $a \in I$ then $C(a) \subseteq I$.
Conversely, since $C(a) \subseteq I$, for all $y \in C(a)$, we have $a+x = y \in I$, for some $x \in I$. By incline property 2.5, we have $a \leq y$ and by Definition 2.11, $a \in I$. □

Remark 4.4
If R is an incline with zero element ‘0’ then C(0) = I.

Remark 4.5
From the Lemma 4.3, we have $I = \cup_{a \in I} C(a)$.
However C(a)’s are not disjoint. Hence C(a)’s does not form a partition of I as in the case of a ring (p.187, Theorem B of [6]) . This is illustrated in the following example:

Example 4.6
Let us consider the incline in Example 3.3.
$I = \{0, b, d\}$ is an ideal of R.
Here, $C(0) = I$, $C(b) = \{b\} \subseteq I$ and $C(d) = \{d, b\} \subseteq I$.
Also, we have $C(b) \cap C(d) = \{b\} \neq \{\ \}$.
Therefore, the set of all cosets does not form a partition of I.

Theorem 4.7
In an incline R, if $a \in R$ is regular, then the coset $C(a) \subseteq a\{1\}$, the set of all 1-inverses of a, and C(a) is a subincline.

Proof
Let y be an arbitrary element of C(a).
That is, $y \in C(a)$ imples, $y = a+x$, for some $x \in I$.
By incline property (2.4), we have $a \leq y$, for all $y \in C(a)$.
Since a is regular by Lemma 2.8, $a = aa \leq ay$ and $a \leq aya \leq a$ which implies $a = aya$, for all $y \in C(a)$.
Hence each element of C(a) is a 1-inverse of a.
Thus, $C(a) \subseteq a\{1\}$.
From Proposition 2.9, we have a is the smallest element of a{1} and from $C(a) \subseteq a\{1\}$, it follows that a is the smallest element of C(a).
Let, $(a+x), (a+y) \in C(a) = a+I$, for some $x, y \in I$.

$$\begin{aligned} \text{We have, } (a+x) + (a+y) &= a + x+y \\ &= a+u \quad (x+y = u \in I) \\ (a+x)(a+y) &= aa+ax+ay+xy \\ &= a + xy \quad \text{(by Lemma 2.8 and Definition 2.1)} \\ &= a+v \quad (xy = v \in I) \end{aligned}$$

Therefore, C(a) is a subincline. □

Remark 4.8
In the above Theorem 4.7, the regularity of ‘a’ is essential. This is illustrated in the following example:

Example 4.9
Let us consider the incline in Example 3.3.
$I = \{0, b, d\}$ is an ideal of R. Here ‘0’ and ‘d’ are the only regular elements.
Hence, $C(0) = I$, $C(d) = \{d, b\}$ are subinclines and $C(b) = \{b\}$ is not a subincline,

here'b' is not regular and b.b = d $\notin$ C(b).

Lemma 4.10
For a $\neq$ b in an incline R, if C(a) = C(b) then a ~ b with respect to I.

Proof
For a $\neq$ b $\in$ R, if C(a) = C(b) then by Definition 4.1,
a+I = b+I. Hence, a + i_1 = b + i_2, for some i_1, i_2 $\in$ I
$\Rightarrow$ a ~ b with respect to I (by Definition 2.12)
$\Rightarrow$ $[a]_I = [b]_I$ (Remark 2.13) □

Remark 4.11
The converse of the Lemma 4.10 is not true. This is illustrated in the following example:

Example 4.12
Let us consider the incline in Example 3.3.
I = {0, b, d} is an ideal of R.
The cosets of R: C(0) = I, C(a) = {a, 1}, C(b) = {b}, C(c) = {c,1},C(d) = {d, b}, C(1) = {1}.
The equivalence classes of R: $[0]_I = [b]_I = [d]_I$ = I and $[a]_I = [c]_I = [1]_I$ = {a, c, 1}.
Here we have b ~ d with respect to I, but C(b) $\neq$ C(d).

Theorem 4.13
For an element 'a' in an incline R and I is an ideal of R, C(a) $\subseteq [a]_I$.

Proof
Let 'b' be an arbitrary element of C(a).
Thus, b$\in$C(a) $\Rightarrow$ b = a+ i_1, for some $i_1 \in$I
$\Rightarrow$ b+i_2 = a+ i_1+ i_2, for some i_1, i_2 $\in$ I
$\Rightarrow$ b+i_2 = a+ i_3, where $i_3 = i_1 + i_2 \in$ I
$\Rightarrow$ b ~ a, by Definition 2.12
$\Rightarrow$ b $\in [a]_I$
$\Rightarrow$ C(a) $\subseteq [a]_I$. □

Remark 4.14
In general, for a$\in$R, C(a) is not a subset of I and I is not a subset of C(a). This is illustrated in following example:

Example 4.15
Let us consider the incline R in Example 4.12 and the ideal I = {0, b, d} of R.
Let 'a' be an element of R. Here C(a) = {a, 1} $\not\subset$ I and I $\not\subset$ C(a).

Theorem 4.16
Let R be an incline and I be an ideal of R.
Then $\Re$(I) = {C(x) / C(x) is the coset of x$\in$R with respect to I}, the set of all cosets with respect to I is an incline with respect to the operations,
(i) C(x) + C(y) = (x+I) + (y+I) = (x+y)+I = C(x+y), for all x, y $\in$ R.
(ii) C(x)C(y) = (x+I)(y+I) = (xy)+I = C(xy), for all x, y $\in$ R.

Proof
Let R be an incline and I be an ideal of R.
Let x+I, y+I, z+I $\in \Re$(I).
1) (x+I) + (y+I) = (x+y +I) = (y+x +I) = (y+I) + (x+I)
2) (x+I) + ((y+I)+(z+I)) = (x+I) + (y+z + I) = (x+y+z + I) = (x+y +I) + (z+I) =

$((x+I)+(y+I)) + (z+I)$
3) $(x+I)((y+I)(z+I)) = (x+I)(yz+I) = (xyz+I) = (xy+I)(z+I) = ((x+I)(y+I))(z+I)$
4) $(x+I)((y+I)+(z+I)) = (x+I)(y+I) + (x+I)(z+I) = (xy+I) + (xz+I)$
5) $((y+I)+(z+I))(x+I) = (y+I)(x+I) + (z+I)(x+I) = (yx+I) + (zx+I)$
6) $(x+I) + (x+I) = (x+x + I) = (x+I)$
7) $(x+I) + ((x+I)(y+I)) = (x+I) + (xy+I) = (x+xy +I) = (x+I)$
8) $(y+I) + ((x+I)(y+I)) = (y+I) + (xy+I) = (y+xy +I) = (y+I)$
Thus, $\Re(I)$ is an incline and we call this incline as the *coset incline*. □

Definition 4.17
$(\Re(I), \leq)$ is a coset incline with order relation '$\leq$' defined as, $C(x) \leq C(y)$ if and only if $C(x) + C(y) = C(y)$, for some $C(x), C(y) \in \Re(I)$.

Theorem 4.18
If I and J are any two ideals of an incline R with zero element, such that $I \subseteq J \subseteq R$, then $J(I) = \{C(a) / C(a)$ is the coset of $a \in J$ with respect to I$\}$ is an ideal of the coset incline $\Re(I)$.

Proof
Let $\Re(I)$ be the coset incline. Since $I \subseteq J$, $J(I)$ is a subset of $\Re(I)$.
To prove J(I) is an ideal of $\Re(I)$, let $C(j_1) = j_1+I$ and $C(j_2) = j_2+I \in J(I)$ with $j_1, j_2 \in J$.
We have, $(j_1+I) + (j_2+I) = (j_1+j_2)+I \in J(I)$ and $(j_1+I)(j_2+I) = (j_1j_2)+I \in J(I)$.
Therefore, J(I) is a subincline of $\Re(I)$.
Since $I \subseteq J$, $0+I = I \in J(I)$, which implies J(I) is non-empty.
If $C(j) \in J(I)$ and $C(r) \leq C(j)$, for some $C(r) \in \Re(I)$, then by Definition (4.17),
we have, $C(r) + C(j) = C(j) \Rightarrow (r+I) + (j+I) = (j+I)$
$\Rightarrow (r+j) + I = j+I$
$\Rightarrow C(r+j) = C(j)$
By Lemma 4.10, we have $r+j \sim j$ with respect to I and by Definition 2.12, there exist $i_1, i_2 \in I$ such that $r+j + i_1 = j + i_2$
$\Rightarrow r + j_1 = j_2$, where $j + i_1= j_1 \in J$ and $j + i_2= j_2 \in J$
$\Rightarrow r \leq j_2$, by incline property (2.4)
$\Rightarrow r \in J$, by Definition (2.11)
Therefore, $r+I \in J(I)$.
Thus, J(I) is an ideal of $\Re(I)$. □

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