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Approximation of signal in the L_p and $Lip(\alpha, p)$-classes by Deferred Cesàro transform

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Abstract: In this article, we determine the degree of approximation of 2π -conjugate periodic signal in the L_p and $Lip(\alpha, p)$ -classes by deferred Cesàro transform.

1. Definitions and Notations: Let s (t) $\in L_p[0,2\pi]$ be a 2π - periodic signal and let its Fourier trigonometric series be given by

$$s(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t), \tag{1.1}$$

and let the conjugate Fourier trigonometric series corresponding to (1.1) is given by

$$\tilde{s} \sim \sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt) \equiv \sum_{n=0}^{\infty} B_n(t), \tag{1.2}$$

Let n^{th} partial sums of (1.1) and (1.2) be given by $s_n(t)$ and $\tilde{s}_n(t)$, respectively.
Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative integers satisfying the followings

$$p_n < q_n, \quad \text{and} \quad \lim_{n\to\infty} q_n = \infty. \tag{1.3}$$

The processor

$$D_n(s_n) = \frac{1}{q_n - p_n}\sum_{k=p_n}^{q_n} s_k(t), \tag{1.4}$$

defines the deferred Cesàro - transform $D(p_n, q_n)$ ([1], also see [3]). It is known [1] that $D(p_n, q_n)$ is regular under conditions (1.3). Note that $D(0,n)$ is the (C, 1) transform with the assumption that $\{\lambda_n\}$ be a monotone non –decreasing sequence of positive integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \le 1$, then $D(n-\lambda_n, n)$ is same as the n^{th} generalized de la Vallée Poussin processor, generated by the sequence $\{\lambda_n\}$[6].

The space $L_p[0,2\pi]$, for p = ∞ includes the spaces of $C_{2\pi}$ of all continuous signals defined over $[0,2\pi]$ (p. 45, [8]).

We write

$$\omega(\delta, s) = \sup_{0\le h\le\delta}|s(t+h) - s(t)|, \tag{1.5}$$

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$$\omega_p(\delta, s) = \sup_{0 \le h \le \delta} \| s(t+h) - s(t) \|_p , \tag{1.6}$$

which are respectively, called the modulus of continuity and the integral modulus of continuity. In particular when $\omega_p(\delta; s) = O(\delta^{\alpha}), (0 < \alpha \le 1)$, then $\omega_p(\delta; s)$ reduces to $Lip(\alpha, p)$. We shall use following notations.

$$\tilde{s}_t(t_1) = \frac{1}{2\pi} \int_0^{\pi} \psi_{t_1}(t) \cot \frac{t}{2} dt , \tag{1.7}$$

$$\psi_{t_1}(t) = \tilde{s}(t_1 + t) - \tilde{s}(t_1 - t). \tag{1.8}$$

2. Known results: Singh and Mahajan [7] established following theorems in L_p and $Lip(\alpha, p)$-norms by (C,1)(E,1)-transform.

Theorem A. Let $\tilde{s} \in L_P(p > 1)$ be a periodic signal and let $\omega_p(s; t)$ satisfies the following condition

$$\int_t^{\pi} \frac{\omega_p(\tilde{s}; u)}{u^2} du = O(H(t)), \tag{2.1}$$

where H(t) $\uparrow$ and H(t) ≥ 0, then we have

$$\| \tilde{t}_n(\tilde{s}; t) - \tilde{s} \|_p = O\left(\frac{1}{n} H\left(\frac{\pi}{n} \right) \right). \tag{2.2}$$

Theorem B. Let $\tilde{s} \in Lip(\alpha, p), 0 < \alpha \le 1, \; p > 1$, then we have

$$\| \tilde{t}_n(\tilde{s}; t) - \tilde{s} \|_p = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1 \\ O\left(\frac{\log n}{n} \right), & \alpha = 1. \end{cases} \tag{2.3}$$

Theorem C. Let $\tilde{s} \in Lip(\alpha, p), \; 0 < \alpha \le 1, p > 1, \alpha p > 1,$

$$| \tilde{t}_n(\tilde{s}; t) - \tilde{s} | = O(n^{-\alpha + \frac{1}{p}}) . \tag{2.4}$$

3. Main results: The object of this paper is to extend the above results for deferred Cesàro – transform. We shall prove following:

Theorem 1. Let $\tilde{s} \in L_P(p > 1)$ be a periodic signal and let $\omega_p(s; t)$ satisfies the follow - ing condition

$$\int_t^{\pi/2} \frac{\omega_p(\tilde{s}; u)}{u^2} du = O(H(t)), \tag{3.1}$$

where H(t) $\uparrow$ and H(t) ≥ 0, then we have

$$\left\| \tilde{D}_n(\tilde{s}_n) - \tilde{s}_{2t}(t_1) \right\|_p = O\{(\pi/2(q_n - p_n))H(\pi/2(q_n - p_n))\}. \tag{3.2}$$

Theorem 2. Let $\tilde{s} \in Lip(\alpha, p), 0 < \alpha \le 1, \; p > 1$, then we have

$$\left\|\tilde{D}_n(\tilde{s}_n)-\tilde{s}_{2t}(t_1)\right\|_p=\begin{cases} O\{(q_n-p_n)^{-\alpha}\}, & 0<\alpha<1 \\ O\{\log(q_n-p_n)/(q_n-p_n)\}, & \alpha=1. \end{cases} \tag{3.3}$$

Theorem 3. Let $\tilde{s}\in Lip(\alpha,p),\ 0<\alpha\le 1,\ p>1,\ \alpha p>1,$

$$\left|\tilde{D}_n(\tilde{s}_n)-\tilde{s}_{2t}(t_1)\right|=O\{(q_n-p_n)^{-\alpha+(1/p)}\}. \tag{3.4}$$

4. Lemmas. We shall use the following lemmas.

Lemma1.([4],p.148). If h(x,t) is a function of two variables defined for $0\le t\le\pi,\ 0\le x\le 2\pi$, then

$$\left\|\int h(x,t)\,dt\right\|_p\le\int\|h(x,t)\|dt \qquad (p>1).$$

Lemma 2. ([5], Theorem 5(ii)). Let $s(t)\in Lip(\alpha,p), 0<\alpha\le 1, p>1, \alpha p>1,$ then s(t) is equivalent to a function $g\in Lip\left(\alpha-\frac{1}{p}\right)$ and $\phi(t)=O\left(t^{\alpha-\frac{1}{p}}\right)$ almost everywhere.

Lemma 3. [2]. If (3.1) hold, then

$$\omega_p(\tilde{s};t)=O(t\,H(t)).$$

Proof of theorem 1. Following Zygmund [8], we have

$$\tilde{D}_n(\tilde{s}_n)-\tilde{s}_t(t_1)=-\frac{1}{(q_n-p_n)\pi}\int_0^{\pi}\frac{\psi_{t_1}(t)}{2\sin(t/2)}\sum_{k=p_n}^{q_n}\cos(k+(1/2))t\ dt$$

$$=-\frac{1}{(q_n-p_n)\pi}\int_0^{\pi}\frac{\psi_{t_1}(t)}{(2\sin(t/2))^2}\sum_{k=p_n}^{q_n}2\sin(t/2)\cos(k+(1/2))t\ dt$$

$$=-\frac{1}{(q_n-p_n)\pi}\int_0^{\pi}\frac{\psi_{t_1}(t)}{(2\sin(t/2))^2}\cos(q_n+p_n+1)(t/2)\sin(q_n-p_n+1)(t/2)\,dt$$

$$\tilde{D}_n(\tilde{s}_n)-\tilde{s}_{2t}(t_1)=-\frac{1}{(q_n-p_n)\pi}\int_0^{\pi/2}\frac{\psi_{t_1}(t)}{(2\sin(t/2))^2}\cos(q_n+p_n+1)(t/2)\sin(q_n-p_n+1)(t/2)\,dt$$

$$\left|\tilde{D}_n(\tilde{s}_n)-\tilde{s}_{2t}(t_1)\right|\le\frac{1}{(q_n-p_n)}\int_0^{\pi/2}\frac{|\psi_{t_1}(2t)|}{(\sin t)^2}\left|\cos(q_n+p_n+1)t\sin(q_n-p_n+1)t\right|dt. \tag{5.1}$$

Now, writing,

$$\int_0^{\pi/2}=\int_0^{\pi/2(q_n-p_n)}+\int_{\pi/2(q_n-p_n)}^{\pi/2}=I_1+I_2,\ say$$

Applying the Minkowski inequality to the right side of (5.1), we get

$$\left\|\tilde{D}_n(\tilde{s}_n)-\tilde{s}_{2t}(t_1)\right\|_p\le\|I_1\|_p+\|I_2\|_p, \tag{5.2}$$

$$\|I_1\|\le\frac{1}{(q_n-p_n)}\left\|\int_0^{\pi/2(q_n-p_n)}\frac{|\psi_{t_1}(2t)|}{(t)^2}\cos(q_n+p_n+1)t\sin(q_n-p_n+1)t\,dt\right\|_p$$

$$=O(1)\frac{1}{(q_n-p_n)}\int_0^{\pi/2(q_n-p_n)}\frac{\|\psi_{t_1}(2t)\|_p}{(t)^2}\left|\cos(q_n+p_n+1)t\sin(q_n-p_n+1)t\right|dt$$

We note that

$$\left\|\psi_{t_1}(2t)\right\|_p = O\{\omega_p(\tilde{s};t)\},$$

thus

$$\begin{aligned}\| I_1 \|_p &= O(1)\ \frac{1}{q_n - p_n} \int_0^{\pi/2(q_n - p_n)} \frac{\omega_p(\tilde{s};t)}{t^2}(q_n - p_n)t\ dt \\ &= O(1) \int_0^{\pi/2(q_n-p_n)} \frac{\omega_p(\tilde{s};t)}{t}\ dt \\ &= O\{(\pi/2(q_n - p_n))H(\pi/2(q_n - p_n))\}. \qquad (5.3)\end{aligned}$$

Again using Lemma 3,

$$\begin{aligned}\| I_2 \|_p &= O(1)\ \frac{1}{(q_n - p_n)} \int_{\pi/2(q_n-p_n)}^{\pi} \frac{\omega_p(\tilde{s};t)}{t^2}\ dt \\ &= O\{(\pi/2(q_n - p_n))H(\pi/2(q_n - p_n))\}. \qquad (5.4)\end{aligned}$$

Combining (5.3) and (5.4), we get (3.2).
This completes the proof of Theorem 1.

Proof of theorem 2. Since $\tilde{s} \in Lip\,(\alpha, p), 0 < \alpha \le 1,\ p > 1,$ therefore

$$\omega_p(\tilde{s};t) = O(t^{\alpha})\text{, and}$$

set

$$H(u) = \begin{cases} u^{\alpha-1}, & 0 < \alpha < 1 \\ \log\dfrac{1}{u}, & \alpha = 1, \end{cases} \qquad (5.5)$$

then by Theorem 1, we get (3.3).
This completes the proof of Theorem 1.

Proof of theorem 3. We have from the proof of Theorem 1 and Lemma 2

$$\begin{aligned}| I_1 | &= O(1)\frac{1}{q_n - p_n} \int_0^{\pi/2(q_n-p_n)} t^{\alpha - \frac{1}{p} - 1}(q_n - p_n)dt \\ &= O\left((q_n - p_n)^{-\alpha + (1/p)}\right). \qquad (5.6)\end{aligned}$$

Again from Lemma 2,

$$\begin{aligned}| I_2 | &= O(1)\frac{1}{q_n - p_n} \int_{\pi/2(q_n-p_n)}^{\pi} t^{\alpha - (1/p) - 2}\ dt \\ &= O\left((q_n - p_n)^{-\alpha + (1/p)}\right) . \qquad (5.7)\end{aligned}$$

Combining (5.6) and (5.7), we get, (3.4)
This completes the proof of Theorem 3.

6. Corollaries: If we put $q_n = n$ *and* $p_n = n - \lambda_n$, then deferred Cesàro - transform reduces to n^{th} generalized de la Vallée Poussin means $V_n(\lambda)$ then from Theorem 1 ,2 and 3 , we get following respectively:

Corollary1. Let $\tilde{s} \in L_P(p > 1)$ be a periodic signal and let $\omega_p(s;t)$ satisfies (3.1), then

$$\left\| V_n(\tilde{s}_n) - \tilde{s}_{2t}(t_1) \right\|_p = O\{(\pi/2\lambda_n) H(\pi/2\lambda_n)\}.$$

Corollary 2 .Let $\tilde{s} \in Lip(\alpha, p), 0 < \alpha \le 1, \ p > 1,$ then we have

$$\left\| V_n(\tilde{s}_n) - \tilde{s}_{2t}(t_1) \right\|_p = \begin{cases} O\{(\lambda_n)^{-\alpha}\}, & 0 < \alpha < 1 \\ O\{\log(\lambda_n)/\lambda_n\}, & \alpha = 1. \end{cases}.$$

Corollary3. Let $\tilde{s} \in Lip(\alpha, p), \ 0 < \alpha \le 1, \ p > 1, \alpha p > 1,$

$$\left| V_n(\tilde{s}_n) - \tilde{s}_{2t}(t_1) \right| = O\{(\lambda_n)^{-\alpha + (1/p)}\}.$$

Competing Interests

The authors declare that they have no competing interests.

Author's contribution

All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

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