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A STUDY OF FRACTIONAL INTEGRAL OPERATORS AND GENERALIZED k-WRIGHT FUNCTION

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ABSTRACT

In this paper we establish two theorems where in we have obtained the image of generalized k-Wright hypergeometric function under the fractional integral operators involving Fox's H-function. Four special cases of these theorems have also been derived. On account of the general nature of our result a large number of new and known results follow as special cases of our main findings.

1. INTRODUCTION:

Generalized k-Gamma function $\Gamma_k(x)$ defined as (Diaz and Pariguan [1])

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{(x/k)-1}}{(x)_{n,k}}, \; k > 0, \; x \in C \setminus kZ^- \tag{1}$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\ldots\ldots\ldots.(x+(n-1)k), \; x \in C, k \in R, n \in N^+ \tag{2}$$

For $\mathrm{Re}(x) > 0$, $\Gamma_k(x)$ is defined as the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\left(\frac{t^k}{k}\right)} dt \tag{3}$$

From equation (3) it follows that

$$\Gamma_k(x) = k^{(x/k)-1}\,\Gamma(x/k) \tag{4}$$

The generalized Wright hypergeometric function [8] for $z, a_i, b_j \in C$ and $\alpha_i, \beta_j \in R$ $(\alpha_i, \beta_j \neq \mathbf{0}\,;\, i = \mathbf{1}, \mathbf{2}, \ldots\ldots\ldots, p : j = \mathbf{1}, \mathbf{2}, \ldots. q)$ will be represented in the following manner:

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\,\frac{z^n}{n\,!} \tag{5}$$

The generalized k- Wright hypergeometric function ${}_p\psi_q{}^k(z)$ is defined by Gehlot and Prajapati [3] for

$z, a_i, b_j \in C, k \in R^+, \alpha_i, \beta_j \in R\ (\alpha_i, \beta_j \neq \mathbf{0}\,;\, i = \mathbf{1}, \mathbf{2}, \ldots\ldots\ldots, p : j = \mathbf{1}, \mathbf{2}, \ldots. q)$

and $(a_i + \alpha_i n)$, $(b_j + \beta_j n) \in C / kZ^-$

$${}_p\psi_q{}^k(z) = {}_p\psi_q{}^k\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)}\,\frac{z^n}{n!} \tag{6}$$

For convergence, we use the following notations

$$\delta = \sum_{j=1}^{q}\left(\frac{\beta_j}{k}\right) - \sum_{i=1}^{p}\left(\frac{\alpha_i}{k}\right)\ ;\ D^{-1} = \prod_{i=1}^{p}\left|\frac{\alpha_i}{k}\right|^{\frac{-\alpha_i}{k}}\prod_{j=1}^{q}\left|\frac{\beta_j}{k}\right|^{\frac{\beta_j}{k}}$$

$$\mu = \sum_{j=\mathbf{1}}^{q}\left(\frac{b_j}{k}\right) - \sum_{i=\mathbf{1}}^{p}\left(\frac{a_i}{k}\right) + \frac{p-q}{\mathbf{2}}$$

1. If $\delta > -\mathbf{1}$, then the series (6) is absolutely convergent for all $z \in C$ and the generalized

   k- Wright hypergeometric function ${}_p\psi_q{}^k(z)$ is an entire function of z.

2. If $\delta = -\mathbf{1}$, then the series (6) is absolutely convergent for all $|\,z\,| < D^{-1}$.

3. If $\delta = -\mathbf{1}$, then the series (6) is absolutely convergent for all $|z| = D^{-\mathbf{1}}, \mathrm{Re}(\mu) > \frac{\mathbf{1}}{\mathbf{2}}$

Special case of equation (6) becomes in the form by taking
$k = 1, p = 0, q = 1, b_1 = 0, \beta_1 = 1, b_2 = 1 + \delta, \beta_2 = \nu$

$$ {}_0\psi_1 \left[\begin{matrix} - \\ (0,1),(1+\delta,\nu) \end{matrix} \middle| -az^{\mu} \right] = J_{\delta}^{\nu}(az^{\mu}) \tag{7} $$

with complex $z, \nu \in C$, known as Wright generalized Bessel function [7, p.19, eq (2.6.10)]

The Fox's H-function or simply H-function was introduced by Charle's Fox [2].This function is defined and represented by means of the following Mellin-Barnes type of contour integral:

$$ H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \middle| \begin{matrix} (c_j,\gamma_j)_{\mathbf{1},P} \\ (d_j,\delta_j)_{\mathbf{1},Q} \end{matrix} \right] \tag{8} $$

$$ = \frac{1}{2\pi i}\int_L \theta(s) z^s \, ds \tag{9} $$

where $i = \sqrt{-1}$, $z \neq 0$ and

$$ \theta(s) = \frac{\prod_{j=1}^{M}\Gamma(d_j - \delta_j s)\prod_{j=1}^{N}\Gamma(1 - c_j + \gamma_j s)}{\prod_{j=M+1}^{Q}\Gamma(1 - d_j + \delta_j s)\prod_{j=N+1}^{P}(c_j - \gamma_j s)} \tag{10} $$

The nature of the contour L in (9), the conditions for convergence of the integral (9), the asymptotic expansion of the H-function and some of its special cases can be referred to in the works of Srivastava,Gupta and Goyal [7] and Mathai and Saxena [5].

The fractional integral operators involving Fox's H-function were defined and studied by Kantesh Gupta[4] and Saxena and Kumbhat [6] in the following form:

$$ R_{x,r}^{\eta,\alpha}[f(x)] = r x^{-\eta - r\alpha - \mathbf{1}} \int_{\mathbf{0}}^{x} t^{\eta}(x^r - t^r)^{\alpha} H_{P,Q}^{M,N}\left[c\left(\frac{t^r}{x^r}\right)^m \left(\mathbf{1} - \frac{t^r}{x^r}\right)^l \middle| \begin{matrix} (c_j,\gamma_j)_{\mathbf{1},P} \\ (d_j,\delta_j)_{\mathbf{1},Q} \end{matrix} \right] f(t)\,dt \tag{11} $$

$$ K_{x,r}^{\delta,\alpha}[f(x)] = r x^{\delta} \int_{x}^{\infty} t^{-\delta - r\alpha - \mathbf{1}}(t^r - x^r)^{\alpha} H_{P,Q}^{M,N}\left[c\left(\frac{x^r}{t^r}\right)^m \left(\mathbf{1} - \frac{x^r}{t^r}\right)^l \middle| \begin{matrix} (c_j,\gamma_j)_{\mathbf{1},P} \\ (d_j,\delta_j)_{\mathbf{1},Q} \end{matrix} \right] f(t)\,dt \tag{12} $$

where M, N, P, Q are positive integers such that $1 \le M \le Q$, $0 \le N \le P$. m, n and r are also positive integers. The (sufficient) conditions of existence of operators (11) and (12) are given below:

$$(i)\left|\arg(c)\right| < \frac{1}{2}\pi\Omega\ ,\ (\Omega > 0)$$

$$\Omega = \sum_{j=1}^{M}\delta_j - \sum_{j=M+1}^{Q}\delta_j + \sum_{j=1}^{N}\gamma_j - \sum_{j=N+1}^{P}\gamma_j$$

$$(ii) 1 \le p, q < \infty; p^{-1} + q^{-1} = 1;$$

$$(iii)\operatorname{Re}\left[\eta + \left(r\,m\frac{d_j}{\delta_j}\right)\right] > -q^{-1};$$

$$(iv)\operatorname{Re}\left[\alpha + \left(r\,l\frac{d_j}{\delta_j}\right)\right] > -q^{-1};$$

$$(v)\operatorname{Re}\left[\alpha + \delta + \left(r\,m\frac{d_j}{\delta_j}\right)\right] > p^{-1}\,(j = 0,1.....m)$$

$$(vi)\ f(x) \in L_p(0,\infty),$$

2. MAIN THEOREMS

THEOREM 1

$$\text{If}\quad Re(\alpha) > 0, \left|\arg(c)\right| < \frac{1}{2}\pi\Omega,\ \Omega > 0,$$

$$Re\left(\frac{\eta}{r} + \frac{\lambda}{kr} + \frac{\mu n}{kr}\right) + m\min_{1 \le j \le M}\operatorname{Re}\left(\frac{dj}{\delta j}\right) > 0\ ,\ \operatorname{Re}(\alpha) + l\min_{1 \le j \le M}\operatorname{Re}\left(\frac{dj}{\delta j}\right) + 1 > 0$$

then for $\delta > -1$, fractional integral operator $R_{x,r}^{\eta,\alpha}$ of generalized k-Wright hypergeometric function ${}_p\psi_q^{\,k}(z)$ is given by

$$R_{x,r}^{\eta,\alpha}\left[x^{\frac{\lambda}{k}-1}\,{}_p\psi_q^{\,k}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|ax^{\frac{\mu}{k}}\right]\right]$$

$$=x^{\frac{\lambda}{k}-1}\,{}_p\psi_q^{\,k}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|ax^{\frac{\mu}{k}}\right]$$

$$H_{P+2,Q+1}^{M,N+2}\left[c\left|\begin{matrix}(c_j,\gamma_j)_{1,N},(-\alpha,l),\{(1-\frac{\eta}{r}-(\frac{\lambda+\mu n}{kr})),m\},(c_j,\gamma_j)_{N+1,P}\\(d_j,\delta_j)_{1,Q},\{(-\alpha-\frac{\eta}{r}-(\frac{\lambda+\mu n}{kr})),l+m\}\end{matrix}\right.\right] \tag{13}$$

PROOF:

Using equation (11), the left hand side of theorem can be written as

$$R_{x,r}^{\eta,\alpha}\left[x^{\frac{\lambda}{k}-1}\,{}_p\psi_q^{\,k}\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|ax^{\frac{\mu}{k}}\right]\right]$$

$$=rx^{-\eta-r\alpha-1}\int_0^x t^{\eta}(x^r-t^r)^{\alpha}H_{P,Q}^{M,N}\left[c\left(\frac{t^r}{x^r}\right)^m\left(1-\frac{t^r}{x^r}\right)^l\middle|\begin{matrix}(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right]t^{\frac{\lambda}{k}-1}$$

$$\quad{}_p\psi_q^{\,k}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|at^{\frac{\mu}{k}}\right]dt$$

Now expressing ${}_p\psi_q^{\,k}(z)$ in summation form and H-Function in contour integral with the help of equations (6) and (9) respectively and then changing the order of integration and summation which is justified under the conditions stated with theorem, we get

$$R_{x,r}^{\eta,\alpha}\left[x^{\frac{\lambda}{k}-1}\,{}_p\psi_q^{\,k}\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|ax^{\frac{\mu}{k}}\right]\right]=$$

$$=rx^{-\eta-r\alpha-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)(a)^n}{\prod_{j=1}^{q}\Gamma_k(bj+\beta_j n)\,n!}\frac{1}{2\pi i}\int_L\theta(s)c^s\left\{\int_0^x(x^r-t^r)^{\alpha}t^{\eta+\frac{(\mu n+\lambda)}{k}-1}\left(\frac{t^r}{x^r}\right)^{ms}\left(1-\frac{t^r}{x^r}\right)^{ls}dt\right\}ds \tag{14}$$

To evaluate the t-integral substituting $\frac{t^r}{x^r} = z, t = xz^{\frac{1}{r}}, dt = \frac{x}{r} z^{\frac{1}{r}-1} dz$ in the eq(14), we get

$$= x^{-\eta-r\alpha-1} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k (a_i + \alpha_i n)(a)^n}{\prod_{j=1}^{q} \Gamma_k (b_j + \beta_j n) n!} \frac{1}{2\pi i} \int_L \theta(s) c^s \left\{ x^{\eta+\frac{\lambda}{k}+\frac{\mu n}{k}+r\alpha} \int_0^1 z^{ms+\frac{\eta}{r}+\frac{\lambda+\mu n}{kr}-1} (1-z)^{\alpha+ls} dz \right\} ds \tag{15}$$

Finally, on evaluating the z integral with the help of Beta function and re-interpreting the result thus obtained in terms of H-function and ${}_p\psi_q^{\,k}(z)$ we get the required result.

THEOREM 2

$$\textit{If} \quad \mathrm{Re}(\alpha) > \mathbf{0}, \left| \arg(c) \right| < \frac{\mathbf{1}}{\mathbf{2}} \pi\Omega,\ \Omega > \mathbf{0},$$

$$\mathrm{Re}\left\{ \frac{(\mathbf{1}+\delta)}{r} - \frac{\lambda}{kr} - \frac{\mu n}{kr} \right\} + m \min_{\mathbf{1} \le j \le M} \mathrm{Re}\left(\frac{d_j}{\delta_j} \right) > \mathbf{0}\ ,\ \mathrm{Re}(\alpha) + l \min_{\mathbf{1} \le j \le M} \mathrm{Re}\left(\frac{d_j}{\delta_j} \right) + \mathbf{1} > \mathbf{0}$$

then

$$K_{x,r}^{\delta,\alpha}\left[x^{\frac{\lambda}{k}-1}\, {}_p\psi_q^{\,k}\left[\left. \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right| a x^{\frac{\mu}{k}} \right] \right]$$

$$= x^{\frac{\lambda}{k}-1}\, {}_p\psi_q^{\,k}\left[\left. \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right| a x^{\frac{\mu}{k}} \right]$$

$$H_{P+2,Q+1}^{M,N+2}\left[c \left| \begin{matrix} (c_j,\gamma_j)_{1,N}, (-\alpha,l), \{(1-\frac{1}{r}(1+\delta-(\frac{\lambda+\mu n}{k}))), m\}, (c_j,\gamma_j)_{N+1,P} \\ (d_j,\delta_j)_{1,Q}, \{(-\alpha-\frac{1}{r}(1+\delta-(\frac{\lambda+\mu n}{k}))), l+m\} \end{matrix} \right. \right] \tag{16}$$

PROOF:

The proof of Theorem 2 can be developed on the lines similar to those given with proof of Theorem1.

3. SPECIAL CASES

1. If in Theorem 1 we reduce H-function involved in $R_{x,r}^{\eta,\alpha}$ to ${}_{\mathbf{1}}F_{\mathbf{0}}$ [7, p.18, eq (2.6.4)], by taking $M=1, N=1, P=1, Q=1, r=1, m=1, l=0, c_1=1-\kappa, \gamma_1=1, d_1=0, \delta_1=1$ we get the following interesting result after a little simplification.

If $$R_{x,1}^{\eta,\alpha}[f(x)] = x^{-\eta-\alpha-1}\int_0^x t^{\eta}(x-t)^{\alpha}\,\Gamma(\kappa)(1+\frac{ct}{x})^{-\kappa} f(t)\,dt \qquad (17)$$

then

$$x^{-\eta-\alpha-1}\int_0^x t^{\eta}(x-t)^{\alpha}(1+\frac{ct}{x})^{-\kappa}\;{}_p\psi_q^{\,k}\left[\left.\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right| at^{\frac{\mu}{k}}\right]dt$$

$$=\frac{\Gamma(\alpha+1)}{\Gamma\kappa}x^{\frac{\lambda}{k}-1}\;{}_p\psi_q^{\,k}\left[\left.\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right| ax^{\frac{\mu}{k}}\right]H_{2,2}^{1,2}\left[c\left|\begin{matrix}(1-\kappa,1),\{(1-(\eta+\frac{\lambda+\mu n}{k})),1\}\\(0,1),\{-(\alpha+\eta+\frac{\lambda+\mu n}{k}),1\}\end{matrix}\right.\right] \qquad (18)$$

The conditions of validity of (18) can be easily derived from the conditions of existence mentioned with the Theorem1

2. Further on taking k=1 in above result the following interesting result is obtained after a little simplification

$$x^{-\eta-\alpha-\mathbf{1}}\int_{\mathbf{0}}^x t^{\eta}(x-t)^{\alpha}(\mathbf{1}+\frac{ct}{x})^{-\kappa}\;{}_p\psi_q\left[\left.\begin{matrix}(a_i,\alpha_i)_{\mathbf{1},p}\\(b_j,\beta_j)_{\mathbf{1},q}\end{matrix}\right| at^{\mu}\right]=\Gamma(\alpha+\mathbf{1})x^{\lambda-\mathbf{1}}$$

$${}_{p+1}\psi_{q+1}\left[\left.\begin{matrix}(a_i,\alpha_i)_{1,p}(\eta+\lambda,\mu)\\(1+\alpha+\eta+\lambda,\mu)(b_j,\beta_j)_{1,q}\end{matrix}\right| ax^{\mu}\right]{}_2F_1[\kappa;\eta+\lambda+\mu n;1+\alpha+\eta+\lambda+\mu n;-c] \qquad (19)$$

The conditions of validity of (19) can be easily follows from conditions mentioned with the Theorem1.

3. In Theorem1 on reducing H-function involved in $R_{x,r}^{\eta,\alpha}$ to Whittaker function [7, p.18, eq(2.6.7)], we get the following interesting result:

If

$$R_{x,1}^{\eta,\alpha}[f(x)] = x^{-\eta-\alpha-1}\int_0^x t^{\eta}(x-t)^{\alpha} e^{-\frac{c}{2}(1-\frac{t}{x})} W_{\rho,\upsilon}\{c\,(1-\frac{t}{x})\}\, f(t)dt \tag{20}$$

then

$$x^{-\eta-\alpha-1}\int_0^x t^{\eta}(x-t)^{\alpha} e^{-\frac{c}{2}(1-\frac{t}{x})} W_{\rho,\upsilon}\{c(1-\frac{t}{x})\}\, {}_p\psi_q{}^{k}\left[\begin{matrix}(a_i,\alpha_i)\\(b_j,\beta_j)\end{matrix}\middle| at^{\frac{\mu}{k}}\right]dt$$

$$= \Gamma(1+\alpha)\, x^{\frac{\lambda}{k}-1}\, {}_p\psi_q{}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| ax^{\frac{\mu}{k}}\right] H_{2,3}^{2,1}\left[c\middle|\begin{matrix}(-\alpha,0),\{(1-(\eta+\frac{\lambda+\mu n}{k})),1\},(1-\rho,1)\\(\upsilon+\frac{1}{2},1),(-\upsilon+\frac{1}{2},1),\{-(\alpha+\eta+\frac{\lambda+\mu n}{k}),1\}\end{matrix}\right] \tag{21}$$

The conditions of validity of above corollary can be easily obtained from the existence conditions of Theorem1

4. On reducing ${}_p\psi_q{}^{k}(z)$ involved in the Theorem 2 to Wright Generalized Bessel function of first kind (7), we get the following result

$$K_{x,r}^{\delta,\alpha}\left[x^{\lambda-1}J_{\delta}^{\nu}(ax^{\mu})\right]$$

$$= x^{\lambda-1}J_{\delta}^{\nu}(ax^{\mu})$$

$$\times\ H_{P+2,Q+1}^{M,N+2}\left[c\middle|\begin{matrix}(c_j,\gamma_j)_{1,N}(-\alpha,l)\{(1-\frac{1}{r}(1+\delta-(\lambda+\mu n))),m\}(c_j,\gamma_j)_{N+1,P}\\(d_j,\delta_j)_{1,Q}\{(-\alpha-\frac{1}{r}(1+\delta-(\lambda+\mu n))),l+m\}\end{matrix}\right] \tag{22}$$

provided that the conditions of existence of the above result follows easily with the help of Theorem 2.

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