LABELING OF SUBDIVIDED GRAPHS

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Abstract

In this paper we prove that the graph obtained by the subdivision of the edges of the star \( S_{1,n} \), the graph obtained by the subdivision of the central edge of the Bistar \( B_{m,n} \), the graph obtained by subdividing the edges of the path \( P_n \) in \( P_n \square K_1 \) and \( P_n \square K_2 \), the graph obtained by subdividing the parallel edges of a H-graph in \( H \square K_1 \) and in \( H \square K_2 \) are Skolem difference mean.

Keywords: Star, Bistar, Path, Subdivision, Labeling, Skolem Difference Mean labeling, Skolem Difference Mean graphs.

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1 Introduction

Throughout this paper, by a graph, we mean a finite, undirected, simple graph. Let \( G (V, E) \) be a graph with \( p \) vertices and \( q \) edges. The Star \( S_{1,n} \) is obtained from \( K_1 \) by joining \( n \) pendant edges with \( K_1 \). The Bistar \( B_{m,n} \) is the graph obtained from \( K_2 \) by joining \( m \) pendant edges to one end of \( K_2 \) and \( n \) pendant edges to the other end of \( K_2 \). The edge of \( K_2 \) is called the central edge of \( B_{m,n} \) and the vertices of \( K_2 \) are called the central vertices of \( B_{m,n} \). A path is a simple graph obtained from a walk \( v_0e_1v_1e_2v_2...e_kv_k \) in which the vertices and edges are distinct. A path on \( n \) vertices is denoted by \( P_n \). The H-graph of a path \( P_n \) is the graph obtained from two copies of \( P_n \) with vertices \( v_1,v_2,...v_n \) and \( u_1,u_2,...u_n \) by joining the vertices \( \frac{v_1+n}{2} \) and \( \frac{u_1+n}{2} \) if \( n \) is odd and the vertices \( \frac{v_1+n}{2} \) and \( \frac{u_1}{2} \) if \( n \) is even [3].

A subdivision of a graph \( G \) is a graph that can be obtained from \( G \) by a sequence of edge subdivisions. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey of many of them can be found in [2]. The concept of Skolem Mean Labeling was introduced in [1].

In this paper, we define Skolem difference mean labeling and discuss the Skolem difference mean labeling of subdivided graphs.

2 Main Results

Definition 2.1 A graph \( G(V,E) \) with \( p \) vertices and \( q \) edges is said to have Skolem Difference Mean labeling if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from \( 1,2,3,...p+q \) in such a way that the edge \( e=uv \) is labeled with

\[ \frac{|f(u) - f(v)|}{2} \text{ if } |f(u)-f(v)| \text{ is even and } \frac{|f(u) - f(v)| + 1}{2} \text{ if } |f(u)-f(v)| \text{ is odd} \]

and the resulting edges get distinct labels from \( 1,2,3,...q \). A graph that admits Skolem difference mean labeling is called Skolem difference mean graph.

The Skolem difference mean labeling of the path \( P_6 \) is given in figure 1.
Theorem 2.2 The graph obtained by the subdivision of the edges of the star $S_{1,n}$ is Skolem difference mean for all values of $n$.

Proof. Let $G$ be the graph obtained by the subdivision of the edges of the star $S_{1,n}$. Let $V(G) = \{u, w_i; 1 \leq i \leq n\}$ and $E(G) = \{uw, w_iu_i; 1 \leq i \leq n\}$

Define the function $f: V(G) \rightarrow \{1, 2, \ldots, 4n+1\}$ by

- $f(u) = 4n+1$
- $f(w_i) = 2i-1; 1 \leq i \leq n$
- $f(u_i) = 2n+2-2i; 1 \leq i \leq n$

Let $f^*$ be the induced edge labeling of $f$. Then

- $f^*(uw_i) = 2n+1-i; 1 \leq i \leq n$
- $f^*(w_iu_i) = n+2-2i; 1 \leq i \leq n$ if $n$ is odd
- $f^*(w_iu_i) = 2[n-2i]-1; 2 \leq i \leq n$ if $n$ is even

Then the induced edge labels are $1, 2, \ldots, 2n$. Hence the theorem.

The Skolem difference mean labeling of the graph obtained by the subdivision of the edges of the star $S_{1,4}$ is given in fig 2.

Theorem 2.3 The graph obtained by the subdivision of the central edge of the Bistar $B_{m,n}$ is a Skolem difference mean graph for all values of $m$ and $n$.

Proof. Let $V(B_{m,n}) = \{u, v, w_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uw, vw, uu_i, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ where $u$ and $v$ are the central vertices and let $w$ be the vertex which subdivides the edge $uv$.

Define the function $f: V(G) \rightarrow \{1, 2, 3, \ldots, 2m+2n+5\}$ by

- $f(u) = 1$
- $f(w) = 2m+2n+5$
- $f(v) = 3$
- $f(u_i) = 2i, 1 \leq i \leq m$
- $f(v_j) = 2m+2n+5-2j, 1 \leq j \leq n$
Let \( f^* \) be the induced edge labeling of \( f \). Then
\[
\begin{align*}
f^*(u_i) &= i; 1 \leq j \leq m \\
f^*(uw) &= m+n+2 \\
f^*(wv) &= m+n+1 \\
f^*(v_j) &= m+n+1-j; 1 \leq j \leq n
\end{align*}
\]
The induced edge labels are 1,2,3,…m+n+2.Hence the theorem.

The Skolem difference mean Labeling of the graph obtained by the subdivision of the central edge of the Bistar \( B_{6,4} \) is given in fig 3.

![Fig 3](image)

**Theorem 2.4** Let \( G \) be a graph obtained from the graph \( P_n \Box K_1 \) by subdividing the edges of the path \( P_n \). Then \( G \) is Skolem difference mean for all values of \( n \).

**Proof.** Let \( V(G) = \{u_i,v_i,w_j; 1 \leq i \leq n, 1 \leq j \leq n-1\} \) and \( E(G) = \{u_iw_i,u_iv_i,w_jw_{i+1}; 1 \leq i \leq n, 1 \leq j \leq n-1\} \)
Define \( f: V(G) \rightarrow \{1,2,...,6n-3\} \) by
\[
\begin{align*}
f(u_i) &= 2i-1, 1 \leq i \leq n \\
f(v_i) &= 2n+2-2i, 1 \leq i \leq n \\
f(w_j) &= 6n-1-2j, 1 \leq j \leq n-1
\end{align*}
\]
Let \( f^* \) be the induced edge labeling of \( f \). Then
\[
\begin{align*}
f^*(u_iv_i) &= n+2-2i; 1 \leq i \leq \frac{n+1}{2} \text{ and} \\
&= 2\left( i-\frac{n+1}{2} \right); \frac{n+1}{2} \leq i \leq n \text{ if } n \text{ is odd} \\
&= n+2-2i; 1 \leq i \leq \frac{n}{2} \text{ and} \\
&= 2\left( i-\frac{n}{2} \right)-1; \frac{n}{2} + 1 \leq i \leq n \text{ if } n \text{ is even.}
\end{align*}
\]
\[
\begin{align*}
f^*(u_iw_j) &= 3n-2j; 1 \leq j \leq n-1 \\
f^*(w_jw_{i+1}) &= 3n-1-2j; 1 \leq j \leq n-1
\end{align*}
\]
Then the induced edge labels are 1,2,…3n-2.Hence the graph \( G \) is Skolem difference mean for all \( n \).

The Skolem difference mean Labeling of the graph obtained by subdividing the edges of the path \( P_5 \) of the graph \( P_5 \Box K_1 \) is given in fig 4.
Theorem 2.5 Let G be a graph obtained from the graph $P_n \square K_2$ by subdividing the edges of the path $P_n$. Then G is Skolem difference mean for all values of n.

Proof. Let $V(G) = \{u_i,v_{i1},v_{i2},w_j; 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and $E(G) = \{u_i v_{i1}, u_i v_{i2}, u_j w_j w_{j+1}; 1 \leq i \leq n, 1 \leq j \leq n-1\}$

Define $f: V(G) \rightarrow \{1,2,\ldots,8n-3\}$ by

- $f(u_i) = 2i-1$, $1 \leq i \leq n$
- $f(v_{i1}) = 6i-4$, $1 \leq i \leq n$
- $f(v_{i2}) = 6i-2$, $1 \leq i \leq n$
- $f(w_j) = 8n-1-2j$, $1 \leq j \leq n-1$

Let $f^*$ be the induced edge labeling of f. Then

- $f^*(u_i v_{i1}) = 2i-1$; $1 \leq i \leq n$
- $f^*(u_i v_{i2}) = 2i$; $1 \leq i \leq n$
- $f^*(u_j w_j) = 4n-2j$; $1 \leq j \leq n-1$
- $f^*(w_j u_{j+1}) = 4n-1-2j$; $1 \leq j \leq n-1$

Then the induced edge labels are 1,2,…4n-2. Hence the graph G is Skolem difference mean for all n.

The Skolem difference mean Labeling of the graph obtained by subdividing the edges of the path $P_5$ of the graph $P_5 \square K_2$ is given in fig 5.

Theorem 2.6 The graph G obtained by the subdivision of the parallel edges of a H-graph in $H \square K_1$ is Skolem difference mean.

Proof. Let $V(G) = \{u_i,u'_i,v_i,v'_i,w_j,t_j; 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and $E(G) = \{u_i u'_i,v_i v'_i, u_i w_j w_{j+1}, v_i t_j, t_j v_{j+1}; 1 \leq i \leq n, 1 \leq j \leq n-1\}$

Define a function $f: V(G) \rightarrow \{1,2,\ldots,12n-5\}$ by

- $f(u_i) = 2i-1$; $1 \leq i \leq n$
- $f(v_i) = 10n-1-2i$; $1 \leq i \leq n$
- $f(u'_i) = 4n+2-2i$; $1 \leq i \leq n$
- $f(v'_i) = 6n-2+2i$; $1 \leq i \leq n$
- $f(w_j) = 12n-3-2j, 1 \leq i \leq n-1$
- $V_{n+1}^{n+1} \cdot \frac{U_{n+1}}{2}$ if n is odd\n- $V_{n+1}^{n+1} \cdot \frac{U_{n+1}}{2}$ if n is even;

$1 \leq i \leq n, 1 \leq j \leq n-1$
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Let \( f(t_j) = 2n-1+2i; \ 1 \leq i \leq n-1 \)

Let \( f^* \) be the induced edge labeling of \( f \). Then

\[
\begin{align*}
f^*(u_iu_i') &= 2n+2-2i; \ 1 \leq i \leq n \\
f^*(v_iv_i') &= 2n+1-2i; \ 1 \leq i \leq n \\
f^*(u_jw_j) &= 6n-1-2j; \ 1 \leq j \leq n-1 \\
f^*(w_ju_{j+1}) &= 6n-2-2j; \ 1 \leq j \leq n-1 \\
f^*(v_jt_j) &= 4n-2j; \ 1 \leq j \leq n-1 \\
f^*(t_jv_{j+1}) &= 4n-1-2j; \ 1 \leq j \leq n-1 \\
f^*(\frac{V_{n+1} \cup U_{n+1}}{2}) &= 4n-1
\end{align*}
\]

Then the induced edge labels are 1, 2, …, 6n-3. Hence the theorem.

The Skolem difference mean labeling of a graph \( G \) obtained by the subdivision of the parallel edges of a H-graph in \( H \square K_1 \) is given in figures 6 and 7.

Fig 6

Fig 7

**Theorem 2.7** The graph \( G \) obtained by the subdivision of the parallel edges of a H-graph in \( H \square K_2 \) is Skolem difference mean.

**Proof.** Let \( V(G) = \{u_iu_i', u_i'u_i'', v_iv_i', v_i''v_i', w_jt_j; 1 \leq i \leq n, 1 \leq j \leq n-1 \} \) and

\[
E(G) = \{u_iu_i', u_i'u_i'', v_iv_i', v_i''v_i', w_jt_j, w_ju_{j+1}, v_jv_{j+1}, \frac{V_{n+1} \cup U_{n+1}}{2} \text{ if } n \text{ is odd} \}
\]
Define a function \( f: V(G) \rightarrow \{1, 2 \ldots 16n-5\} \) by

\[
\begin{align*}
    f(u_i) &= 2i - 1; \quad 1 \leq i \leq n \\
    f(u_i') &= 8n + 7 - 6i; \quad 1 \leq i \leq n \\
    f(u_i'') &= 8n + 5 - 6i; \quad 1 \leq i \leq n \\
    f(v_i) &= 14n - 1 - 2i; \quad 1 \leq i \leq n \\
    f(v_i') &= 6n + 4 + 6i; \quad 1 \leq i \leq n \\
    f(v_i'') &= 6n - 2 + 6i; \quad 1 \leq i \leq n \\
    f(w_j) &= 16n - 3 - 2j; \quad 1 \leq j \leq n - 1 \\
    f(t_j) &= 2n - 1 + 2j; \quad 1 \leq j \leq n - 1
\end{align*}
\]

Let \( f^* \) be the induced edge labeling of \( f \). Then

\[
\begin{align*}
    f^*(u_iu_i') &= 4n + 4 - 4i; \quad 1 \leq i \leq n \\
    f^*(u_iu_i'') &= 4n + 3 - 4i; \quad 1 \leq i \leq n \\
    f^*(v_iv_i') &= 4n + 2 - 4i; \quad 1 \leq i \leq n \\
    f^*(v_iw_i) &= 8n + 2 - 4i; \quad 1 \leq i \leq n \\
    f^*(v_iv_i') &= 8n + 3 - 4i; \quad 1 \leq i \leq n \\
    f^*(v_iw_i) &= 8n + 4 - 4i; \quad 1 \leq i \leq n \\
    f^*(w_ju_j) &= 16n - 3 - 2j; \quad 1 \leq j \leq n - 1 \\
    f^*(w_ju_j') &= 16n - 4 - 2j; \quad 1 \leq j \leq n - 1 \\
    f^*(t_jv_j) &= 6n - 2j; \quad 1 \leq j \leq n - 1 \\
    f^*(t_jv_j') &= 6n - 1 - 2j; \quad 1 \leq j \leq n - 1 \\
\end{align*}
\]

\[
\begin{align*}
    f^*(V_{n+1}/U_{n+1}) &= f^*(\bar{V}_{n}/\bar{U}_{n}) = 6n - 1
\end{align*}
\]

Then the induced edge labels are 1, 2, ..., 8n - 3. Hence the theorem.

The Skolem difference mean labeling of a graph \( G \) obtained by the subdivision of the parallel edges of a H-graph in \( H \boxplus K_2 \) is given in figures 8 and 9.

Fig 8
REFERENCES