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Optimal design of N-Policy batch arrival queueing system with server's single vacation, setup time, second optional service and break down

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ABSTRACT

This paper deals with a $M^{[X]}/(G_1,G_2)/1$ queueing system in which all the customers undergo first essential service (FES) and only some of them receive second optional service(SOS) by the same server. In addition to this the server is unreliable and hence subjected to random breakdowns while in service and the server leaves for single vacation when the system is empty. After returning from vacation if there are N or more customers in the system then the server does setup work before it starts the service. Explicit analytical expressions for various performance measures are derived. A cost model for the optimal operating N- Policy that minimizes the total expected cost per unit time is determined and numerical analysis is carried out.

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Introduction

In day-to-day life, one encounters numerous examples of queueing situations, where all arriving customers require the main service and only some may require the subsidiary service provided by the server. K.C.Madhan [1] has done some initial work on the steady state behaviour of M/G/1 queue with second optional service and later Choudhry and Paul [2] extended the results of Madhan [1] to a batch arrival queue under N-policy. The authors mentioned above have focused on reliable servers. Queueing models with second optional service and breakdowns accommodate real world situations more closely. Therefore, it would be practical to consider the N-policy for the batch arrival $M^{[X]}/G/1$ queueing system in which the service is unreliable and all the arriving customers demand the first essential service (FES) where as only some of them demand the second optional service (SOS).

There are several vacation policies and this paper deals with single vacation policy. Also the server needs a random amount of time for preparatory work which is termed as server's setup time (or) startup time. In this model, it is assumed that after returning from vacation if the server finds N (or) more customers in the system then the server is turned on for the start up work of random length D and as soon as the server finishes the setup work, the busy period initiates.

The steady state behavior of queue size distribution is analyzed for this model, using the supplementary variable technique. Various performance measures such as expected system size and expected length of the cycle are also calculated. The PGF of the system size distribution at an arbitrary epoch is derived. The total expected cost function per unit time is derived and the optimal value Ns* which minimize the long – run average cost under linear cost structure is developed for this model. Numerical analysis is also presented to justify the measures calculated for the model.

Model Description

The $M^{[X]}/(G_1, G_2)/1$ Queueing System under consideration has the following specification.

Compound Arrival Process

The customers are assumed to arrive in batches according to compound Poisson process with arrival rate λ. The number of units arrive at an arbitrary instant is a random variable X whose probability distribution is given by $Pr(X = k) = g_k$, k = 1, 2, 3 …

N-Policy Setup Time and Single Vacation

A cycle begins right after the system becomes empty and the server leaves the system for vacation. After returning from vacation, if the server finds N (or) more customers present in the system then the server takes random amount of setup time for preparatory work before starting the service. The setup time is a random variable with finite moments which has the general distribution D(t) and density function d(t). The customers arriving during the vacation period and setup period will join the queue and wait for their turns. If the server returning from vacation finds less than N customers in the system then he stays idle in the system (i.e.) the server takes only single vacation. The period during which the server remains idle in the system to start the preparatory work after returning from vacation is called build up period. The vacation time follows the general distribution V(t) with finite moments and density function v(t).

Busy Period and Server's Breakdown

Immediately after the setup time the busy period starts and customers are served one by one according to FCFS queue discipline. During busy period the server provides each unit two types of heterogeneous services of which, one is optional. (i.e.) the server provides first essential service (FES) to all the arriving customers and after the completion of FES the customers may leave the system with probability (1 – r) (or) may opt for the SOS with probability r ($0 \le r \le 1$). During the services (FES (or) SOS) the server may undergo breakdowns according to the Poisson process with rates a_i, i = 1, 2. Whenever the breakdowns occur the server is sent immediately for repair and the repair times follow the general distributions $U_i(t)$, i = 1, 2 with finite mean and variance. Once the server gets repaired, he is sent back to the service facility to resume the service. Thus the vacation period, setup period, busy period and break down periods constitute a cycle. It is also assumed that the arrival processes, vacation time, service time and setup time are independent of each other. The model described above is depicted in figure I.

Steady State System Size Equations

To obtain the steady state system size equations of the model using supplementary variable technique, we employ the remaining service time, the remaining setup time and the remaining vacation time of the server as the supplementary variables. The following notations and probabilities are used to derive the steady state equations of the model.

N	-	threshold
λ	-	group arrival rate
X	-	arrival size random variable
g_k	-	$Pr(X = k)$, k = 1,2,3…

α_k (h_k) - probability that k customers arrive during a vacation (setup time) , k = 0, 1, 2 …
X(z) - the probability generating function (PGF) of X
$g_k^{(i)}$ - i-fold convolution of $\{g_k\}$ with itself and $g_0^{(0)} = 1$
a_i - Poisson breakdown rate corresponding to the i^{th} phase of service, i = 1, 2
U_i - repair time corresponding to the breakdown a_i.
N(t) - the system size including one in service at time t.

Let the abbreviations C.D.F, p.d.f, LST respectively denote cumulative density function, probability density function, Laplace Stieltjes transform of the random variables (R.V.).

	R.V	C.D.F	p.d.f	LST	Remaining time of the R.V. at t
FES	S_1	$S_1(t)$	$s_1(t)$	$S_1^*(\theta)$	$S_1^0(t)$
SOS	S_2	$S_2(t)$	$s_2(t)$	$S_2^*(\theta)$	$S_2^0(t)$
Vacation time	V	V(t)	v(t)	$V^*(\theta)$	$V^0(t)$
Setup time	D	D(t)	d(t)	$D^*(\theta)$	$D^0(t)$
Repair time	U_i	$U_i(t)$	$u_i(t)$	$U_i^*(\theta)$	$U_i^0(t)$, i = 1, 2

The server's states are denoted by the random variable

$$Y(t) = \begin{cases} -1, & \text{if the server is idle during buildup period} \\ 0, & \text{if the server is on vacation} \\ 1, & \text{if the server is busy with FES} \\ 2, & \text{if the server is down with a customer in FES} \\ 3, & \text{if the server is busy with SOS} \\ 4, & \text{if the server is down with the customer in SOS} \\ 5, & \text{if the server is in the setup state at time t} \end{cases}$$

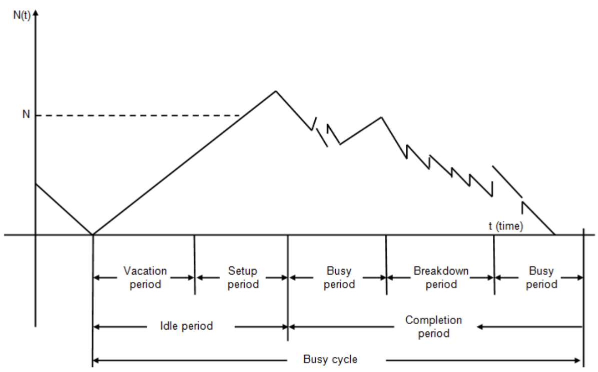

FIGURE I

The state of the system at time t can be described by the Markov process.

$K(t) = \{(Y(t), N(t), S_i^0(t), U_i^0(t), V^0(t), D^0(t)), t \geq 0, i = 1, 2\}$

The transient state system size probabilities are defined by :

$R_n(t) = Pr(Y(t) = -1, N(t) = n), n = 0 \text{ to } N - 1$

$Q_n(x,t)\,dt = Pr\{Y(t) = 0 ; N(t) = n, x \leq V^0(t) \leq x + dt\}, \quad n \geq 0$

$P_{n1}(x, t)\,dt = Pr(N(t) = n, x \leq S_1^0(t) \leq x + dt, Y(t) = 1), \quad n \geq 1$

$P_{n2}(x, t)\,dt = Pr(N(t) = n, x \leq S_2^0(t) \leq x + dt, Y(t) = 3), \quad n \geq 1$

$B_{n1}(x, y, t)\,dt = Pr(N(t) = n, S_1^0(t) = x, y \leq U_1^0(t) \leq y + dt, Y(t) = 2), \quad n \geq 1$

$B_{n2}(x, y, t)\,dt = Pr(N(t) = n, S_2^0(t) = x, y \leq U_2^0(t) \leq y + dt, Y(t) = 4), \quad n \geq 1$

$D_n(x, t)\,dt = Pr(N(t) = n, x \leq D^0(t) \leq x + dt, Y(t) = 5), \quad n \geq N$

Steady State Equations

Under the steady state, the system size probabilities are assumed to be independent of time and the steady state equations are given by,

$$\lambda R_0 = Q_0(0) \tag{1}$$

$$\lambda R_n = Q_n(0) + \lambda \sum_{k=1}^{n} R_{n-k}\, g_k, \qquad 1 \leq n \leq N - 1 \tag{2}$$

$$\frac{-d}{dx} P_{11}(x) = -(\lambda + a_1) P_{11}(x) + (1 - r) P_{21}(0) s_1(x) + B_{11}(x, 0) + P_{22}(0) s_1(x) \tag{3}$$

$$\frac{-d}{dx} P_{n1}(x) = -(\lambda + a_1) P_{n1}(x) + (1 - r) P_{n+11}(0) s_1(x) + B_{n1}(x, 0) + P_{n+12}(0) s_1(x) + \lambda \sum_{k=1}^{n-1} P_{n-k1}(x)\, g_k, \quad 2 \leq n \leq N - 1 \tag{4}$$

$$\frac{-d}{dx} P_{n1}(x) = -(\lambda + a_1) P_{n1}(x) + (1 - r) P_{n+11}(0) s_1(x) + B_{n1}(x, 0) + D_n(0) s_1(x) + P_{n+12}(0) s_1(x) + \lambda \sum_{k=1}^{n-1} P_{n-k1}(x)\, g_k, \quad n \geq N \tag{5}$$

$$\frac{-d}{dx} P_{12}(x) = -(\lambda + a_2) P_{12}(x) + r P_{11}(0) s_2(x) + B_{12}(x, 0) \tag{6}$$

$$\frac{-d}{dx} P_{n2}(x) = -(\lambda + a_2) P_{n2}(x) + r P_{n1}(0) s_2(x) + \lambda \sum_{k=1}^{n-1} P_{n-k2}(x)\, g_k + B_{n2}(x, 0), \quad n \geq 2 \tag{7}$$

$$\frac{-d}{dx} Q_0(x) = -\lambda Q_0(x) + (P_{11}(0)(1 - r) + P_{12}(0)) v(x) \tag{8}$$

$$\frac{-d}{dx} Q_n(x) = -\lambda Q_n(x) + \lambda \sum_{k=1}^{n} Q_{n-k}(x)\, g_k, \qquad n \geq 1 \tag{9}$$

$$\frac{-d}{dx} D_N(x) = -\lambda D_N(x) + Q_N(0)\, d(x) + \lambda \sum_{k=1}^{N} R_{N-k}\, g_k\, d(x), \tag{10}$$

$$\frac{-d}{dx} D_n(x) = -\lambda D_n(x) + Q_n(0)\, d(x) + \lambda \sum_{k=n-N+1}^{n} R_{n-k}\, g_k\, d(x) + \lambda \sum_{k=1}^{n-N} D_{n-k}\, g_k, \quad n \geq N+1 \tag{11}$$

$$\frac{-d}{dy} B_{11}(x, y) = -\lambda B_{11}(x, y) + a_1 P_{11}(x)\, u_1(y) \tag{12}$$

$$\frac{-d}{dy} B_{n1}(x, y) = -\lambda B_{n1}(x, y) + a_1 P_{n1}(x)\, u_1(y) + \lambda \sum_{k=1}^{n-1} B_{n-k1}(x, y)\, g_k, \quad n \geq 2 \tag{13}$$

$$\frac{-d}{dy} B_{12}(x, y) = -\lambda B_{12}(x, y) + a_2 P_{12}(x)\, u_2(y) \tag{14}$$

$$\frac{-d}{dy} B_{n2}(x, y) = -\lambda B_{n2}(x, y) + a_2 P_{n2}(x)\, u_2(y) + \lambda \sum_{k=1}^{n-1} B_{n-k2}(x, y)\, g_k, \quad n \geq 2 \tag{15}$$

LST definition

The following Laplace Stieltjes Transform (LST) are defined to derive the PGF of the system size

$$P^*_{ni}(\theta) = \int_0^\infty e^{-\theta x} P_{ni}(x)\, dx; \qquad S^*_i(\theta) = \int_0^\infty e^{-\theta x}\, d S_i(x), \ i = 1, 2$$

$$B^*_{ni}(\theta, y) = \int_0^\infty e^{-\theta x} B_{ni}(x, y)\, dx, \ i = 1, 2$$

$$D^*_n(\theta) = \int_0^\infty e^{-\theta x} D_n(x)\, dx; \qquad D^*(\theta) = \int_0^\infty e^{-\theta x}\, d D(x)$$

$$Q^*_n(\theta) = \int_0^\infty e^{-\theta x} Q_n(x)\, dx; \qquad V^*(\theta) = \int_0^\infty e^{-\theta x}\, d V(x)$$

Taking the LST on both sides of equations (3) to (15) and using the properties of LST of differentiation, we have,

$$\theta P^*_{11}(\theta) - P_{11}(0) = (\lambda + a_1) P^*_{11}(\theta) - (1 - r) P_{21}(0)\, S^*_1(\theta) - B^*_{11}(\theta, 0) - P_{22}(0)\, S^*_1(\theta) \tag{16}$$

$$\theta P^*_{n1}(\theta) - P_{n1}(0) = (\lambda + a_1) P^*_{n1}(\theta) - (1 - r) P_{n+11}(0)\, S^*_1(\theta) - B^*_{n1}(\theta, 0) - P_{n+12}(0)\, S^*_1(\theta) - \lambda \sum_{k=1}^{n-1} P^*_{n-k1}(\theta)\, g_k, \ 2 \leq n \leq N-1 \tag{17}$$

$$\theta P^*_{n1}(\theta) - P_{n1}(0) = (\lambda + a_1) P^*_{n1}(\theta) - (1 - r) P_{n+11}(0)\, S^*_1(\theta) - B^*_{n1}(\theta, 0) - D_n(0)\, S^*_1(\theta) - P_{n+12}(0)\, S^*_1(\theta) - \lambda \sum_{k=1}^{n-1} P^*_{n-k1}(\theta)\, g_k, \quad n \geq N \tag{18}$$

$$\theta P^*_{12}(\theta) - P_{12}(0) = (\lambda + a_2) P^*_{12}(\theta) - r P_{11}(0)\, S^*_2(\theta) - B^*_{12}(\theta, 0) \tag{19}$$

$$\theta\, P^*_{n2}(\theta) - P_{n2}(0) = (\lambda + a_2)\, P^*_{n2}(\theta) - r\, P_{n1}(0)\, S^*_2(\theta) - \lambda \sum_{k=1}^{n-1} P^*_{n-k2}(\theta)\, g_k - B^*_{n2}(\theta, 0),\ n \geq 2 \tag{20}$$

$$\theta\, Q^*_0(\theta) - Q_0(0) = \lambda\, Q^*_0(\theta) - (P_{11}(0)\,(1 - r) + P_{12}(0))\, V^*(\theta) \tag{21}$$

$$\theta\, Q^*_n(\theta) - Q_n(0) = \lambda\, Q^*_n(\theta) - \lambda \sum_{k=1}^{n} Q^*_{n-k}(\theta)\, g_k,\ n \geq 1 \tag{22}$$

$$\theta\, D^*_N(\theta) - D_N(0) = \lambda\, D^*_N(\theta) - Q_N(0)\, D^*(\theta) - \lambda \sum_{k=1}^{N} R_{N-k}\, g_k\, D^*(\theta) \tag{23}$$

$$\theta\, D^*_n(\theta) - D_n(0) = \lambda\, D^*_n(\theta) - Q_n(0)\, D^*(\theta) - \lambda \sum_{k=n-N+1}^{n} R_{n-k}\, g_k\, D^*(\theta) - \lambda \sum_{k=1}^{n-N} D^*_{n-k}(\theta)\, g_k,\quad n \geq N + 1 \tag{24}$$

$$\frac{-d}{dy} B^*_{11}(\theta, y) = -\lambda\, B^*_{11}(\theta, y) + a_1\, P^*_{11}(\theta)\, u_1(y) \tag{25}$$

$$\frac{-d}{dy} B^*_{n1}(\theta, y) = -\lambda\, B^*_{n1}(\theta, y) + a_1\, P^*_{n1}(\theta)\, u_1(y) + \lambda \sum_{k=1}^{n-1} B^*_{n-k1}(\theta, y)\, g_k,\ n \geq 2 \tag{26}$$

$$\frac{-d}{dy} B^*_{12}(\theta, y) = -\lambda\, B^*_{12}(\theta, y) + a_2\, P^*_{12}(\theta)\, u_2(y) \tag{27}$$

$$\frac{-d}{dy} B^*_{n2}(\theta, y) = -\lambda\, B^*_{n2}(\theta, y) + a_2\, P^*_{n2}(\theta)\, u_2(y) + \lambda \sum_{k=1}^{n-1} B^*_{n-k2}(\theta, y)\, g_k,\ n \geq 2 \tag{28}$$

The LST with respect to repair time are defined by,

$$B^{**1}_{ni}(\theta, \theta_1) = \int_0^\infty e^{-\theta_1 y}\, B^*_{ni}(\theta, y)\, dy$$

$$U^{*1}_i(\theta_1) = \int_0^\infty e^{-\theta_1 y}\, u_i(y)\, dy$$

Taking the LST on both sides of equations (25) to (28) we have,

$$\theta_1\, B^{**1}_{11}(\theta, \theta_1) - B^*_{11}(\theta, 0) = \lambda\, B^{**1}_{11}(\theta, \theta_1) - a_1\, P^*_{11}(\theta)\, U^{*1}_1(\theta_1) \tag{29}$$

$$\theta_1\, B^{**1}_{n1}(\theta, \theta_1) - B^*_{n1}(\theta, 0) = \lambda\, B^{**1}_{n1}(\theta, \theta_1) - a_1\, P^*_{n1}(\theta)\, U^{*1}_1(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k1}(\theta, \theta_1)\, g_k,\quad n \geq 2 \tag{30}$$

$$\theta_1\, B^{**1}_{12}(\theta, \theta_1) - B^*_{12}(\theta, 0) = \lambda\, B^{**1}_{12}(\theta, \theta_1) - a_2\, P^*_{12}(\theta)\, U^{*1}_2(\theta_1) \tag{31}$$

$$\theta_1\, B^{**1}_{n2}(\theta, \theta_1) - B^*_{n2}(\theta, 0) = \lambda\, B^{**1}_{n2}(\theta, \theta_1) - a_2\, P^*_{n2}(\theta)\, U^{*1}_2(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k2}(\theta, \theta_1)\, g_k,\quad n \geq 2 \tag{32}$$

PGF of the system size probabilities

The following partial PGFs for $|z| \le 1$ are defined to determine the system size distribution.

$$R(z) = \sum_{n=0}^{N-1} R_n z^n$$

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{ni}^*(\theta) z^n, \quad P_i(z,0) = \sum_{n=1}^{\infty} P_{ni}(0) z^n, \; i = 1, 2$$

$$D^*(z,\theta) = \sum_{n=N}^{\infty} D_n^*(\theta) z^n, \quad D(z,0) = \sum_{n=N}^{\infty} D_n(0) z^n$$

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta) z^n, \quad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

$$B_i^{**1}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} B_{ni}^{**1}(\theta,\theta_1) z^n$$

$$B_i^*(z,\theta,0) = \sum_{n=1}^{\infty} B_{ni}^*(\theta,0) z^n, \; i = 1, 2$$

Identities

Here some important identities used in this paper are listed out.

- $$\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n-1} P_{n-k2}^*(\theta) g_k\right) = \left(\sum_{n=1}^{\infty} P_{n2}^*(\theta) z^n\right)\left(\sum_{k=1}^{\infty} g_k z^k\right) = P_2^*(z,\theta)\, X(z)$$

- $$\sum_{n=2}^{\infty} z^n \left(\sum_{k=1}^{n-1} P_{n-k1}^*(\theta) g_k\right) = \left(\sum_{n=1}^{\infty} P_{n1}^*(\theta) z^n\right)\left(\sum_{k=1}^{\infty} g_k z^k\right) = P_1^*(z,\theta)\, X(z)$$

- $$\sum_{n=2}^{\infty} z^n \left(\sum_{k=1}^{n-1} B_{n-k1}^{**1}(\theta,\theta_1) g_k\right) = \left(\sum_{n=1}^{\infty} B_{n1}^{**1}(\theta,\theta_1) z^n\right)\left(\sum_{k=1}^{\infty} g_k z^k\right) = B_1^{**1}(z,\theta,\theta_1)\, X(z)$$

- $$\sum_{n=N}^{\infty} z^n \left(\sum_{k=n-N+1}^{n} R_{n-k}\, g_k\right) + \sum_{n=1}^{N-1} z^n \left(\sum_{k=1}^{n} R_{n-k}\, g_k\right) = \left(\sum_{n=0}^{N-1} R_n z^n\right)\left(\sum_{k=1}^{\infty} g_k z^k\right) = R(z)\, X(z)$$

- $$\frac{d}{dz}\left(\frac{1-D^*(W_X(z))\,V^*(W_X(z))}{1-X(z)}\right)_{z=1} = \frac{\lambda^2 E(X)[E(D^2)+2E(D)E(V)+E(V^2)]}{2}$$

- $$\frac{d}{dz}\left(\frac{D^*(W_X(z))(1-V^*(W_X(z)))}{W_X(z)}\right)_{z=1} = \lambda E(X)\left(\frac{E(V^2)}{2}+E(V)E(D)\right)$$

$$* \quad \frac{d}{dz}\left(\frac{1-D^*(W_X(z))}{W_X(z)}\right)_{z=1} = \frac{\lambda\, E(X)\, E(D^2)}{2}$$

Steady state solutions

The closed form expressions after extensive simplification for $D^*(z, \theta)$ and $Q^*(z, \theta)$ are as follows.

$$Q^*(z, \theta) = \frac{P_1(0)\,(V^*(W_X(z)) - V^*(\theta))}{\theta - W_X(z)} \tag{33}$$

$$D^*(z, \theta) = \frac{(D^*(W_X(z)) - D^*(\theta))}{\theta - W_X(z)}\,(P_1(0)\, V^*(W_X(z)) - R(z)\, W_X(z)) \tag{34}$$

Using the equations (19) and (20) and (29) to (32), we get

$$B_i^*(z, \theta, 0) = a_i\, P_i^*(z, \theta)\, U_i^{*1}(W_X(z)) \tag{35}$$

$$B_i^{**1}(z, \theta, \theta_1) = \frac{a_i\, P_i^*(z, \theta)\,(U_i^{*1}(W_X(z)) - U_i^{*1}(\theta_1))}{\theta_1 - W_X(z)}, \; i = 1, 2 \tag{36}$$

$$P_1^*(z, \theta) = \frac{z(P_1(0)\,(D^*(W_X(z))\, V^*(W_X(z)) - 1) - D^*(W_X(z))\, R(z)\, W_X(z))\;(H_{a_1}^*(W_X(z)) - S_1^*(\theta))}{(z - H^*(z))\,(\theta - h_{a_1}(W_X(z)))} \tag{37}$$

$$P_2^*(z, \theta) = \frac{r\, z\, H_{a_1}^*(W_X(z))\,(P_1(0)\,(D^*(W_X(z))\, V^*(W_X(z)) - 1) - D^*(W_X(z))\, R(z)\, W_X(z))\;(H_{a_2}^*(W_X(z)) - S_2^*(\theta))}{(z - H^*(z))\,(\theta - h_{a_2}(W_X(z)))} \tag{38}$$

where $P_1(0) = P_{11}(0)\,(1 - r) + P_{12}(0)$, $W_X(z) = \lambda\,(1 - X(z))$,

$H^*(z) = H_{a_1}^*(W_X(z))\,((1 - r) + r\, H_{a_2}^*(W_X(z)))$, $\quad H_{a_i}^*(W_X(z)) = S_i^*(h_{a_i}(W_X(z)))$ and

$h_{a_i}(W_X(z)) = W_X(z) + a_i(1 - U_i^{*1}(W_X(z)))$, $\quad i = 1, 2$

Then equations (33), (34),(36), (37) and (38) at $\theta = \theta_1 = 0$ respectively give,

$$Q^*(z, 0) = \frac{(1 - V^*(W_X(z)))\, P_1(0)}{W_X(z)}$$

$$D^*(z, 0) = \frac{(1 - D^*(W_X(z)))\,(P_1(0)\, V^*(W_X(z)) - R(z)\, W_X(z))}{W_X(z)}$$

$$B_i^{**1}(z,0,0) = \frac{a_i P_i^*(z,0)(1-U_i^{*1}(W_X(z)))}{W_X(z)}, \; i = 1, 2$$

$$P_1^*(z,0) = \frac{z(P_1(0)(D^*(W_X(z))V^*(W_X(z))-1)-D^*(W_X(z))R(z)W_X(z))\,(1-H_{a_1}^*(W_X(z)))}{(z-H^*(z))(h_{a_1}(W_X(z)))}$$

$$P_2^*(z,0) = \frac{r\,z\,H_{a_1}^*(W_X(z))(P_1(0)(D^*(W_X(z))V^*(W_X(z))-1) - D^*(W_X(z))R(z)W_X(z))\;(1-H_{a_2}^*(W_X(z)))}{h_{a_2}(W_X(z))(z-H^*(z))}$$

Let $P_B(z) = \sum_{i=1}^{2} (P_i^*(z,0) + B_i^{**1}(z,0,0))$

Then $P_B(z) = \dfrac{-z\,\phi(z)(1-H^*(z))}{W_X(z)(z-H^*(z))}$

where $\phi(z) = P_1(0)\,((1 - D^*(W_X(z))\,V^*(W_X(z))) + D^*(W_X(z))\,R(z)\,W_X(z))$

Let $P_I(z)$ gives the PGF of the system size probabilities when the server is idle.
Then $P_I(z) = Q^*(z, 0) + D^*(z, 0) + R(z)$

$$P_I(z) = \frac{(1-V^*(W_X(z)))P_1(0)}{W_X(z)} + \frac{(1-D^*(W_X(z)))(P_1(0)V^*(W_X(z))-R(z)W_X(z))}{W_X(z)} + R(z)$$

To calculate R(z),for $0 \le n \le N-1$, let $\Pi_0 = 1$ and $\Pi_n = \sum_{i=1}^{n} g_i\,\Pi_{n-i}$; $\Psi_0 = \alpha_0$ and $\Psi_n = \sum_{i=0}^{n} \alpha_i\,\Pi_{n-i}$

Using equations(1) and (2) we get

$$R(z) = \frac{P_1(0)}{\lambda}\sum_{n=0}^{N-1} \psi_n z^n = P_1(0)\,\Psi(z), \qquad \text{where } \Psi(z) = \frac{1}{\lambda}\sum_{n=0}^{N-1} \psi_n$$

Then $P_I(z) = \dfrac{\phi(z)}{W_X(z)}$

If P(z) denotes the total probability generating function of the number of customers in the system in steady state, then,

$$P(z) = P_B(z) + P_I(z).$$

$$= \frac{\phi(z)(z-1)H^*(z)}{W_X(z)(z-H^*(z))},$$ where $\phi(z)$ involves the unknown $P_1(0)$ and this can be calculated using the normalizing condition $P(1) = 1$

And $$P_1(0) = \frac{1-\rho_c}{E(D)+E(V)+\sum_{n=0}^{N-1}\frac{\psi_n}{\lambda}}$$

where $\rho_c = \lambda\, E(X)\, E(H_c)$, $E(H_c) = E(S_1)\,(1 + a_1 E(U_1)) + r\,E(S_2)\,(1 + a_2 E(U_2))$

Substituting for $P_1(0)$ in P(z) we have,

$$P(z) = \frac{(1-\rho_c)(z-1)H^*(z)}{z-H^*(z)} \left[\frac{\dfrac{1-D^*(W_X(z))\,V^*(W_X(z))}{W_X(z)} + D^*(W_X(z)) \sum_{n=0}^{N-1} \dfrac{\psi_n}{\lambda} z^n}{E(D)+E(V)+\sum_{n=0}^{N-1} \dfrac{\psi_n}{\lambda}} \right]$$

Performance Measures

In this section, the probability that the server is on vacation (P_V), in busy period (P_{Busy}), in breakdown state (P_{Br}) and in setup state (P_D) are calculated.

i. P_V = the probability that the server is on vacation
= $E(V)\,P_1(0)$, where $P_1(0) = P_{11}(0)\,(1-r) + P_{12}(0)$

ii. P_{Busy} = the probability that the server is busy
= $\lambda\,E(X)\,(E(S_1) + r\,E(S_2)) = \rho_{Busy}$,

iii. P_{Br} = the probability that the server is in break down state
= $\underset{z\to 1}{lt}\,(B_1^*(z,0,0) + B_2^*(z,0,0))$

P_{Br} = $\lambda\,E(X)\,(E(S_1)\,a_1\,E(U_1) + r\,E(S_2)\,a_2\,E(U_2)) = \rho_{Br}$

Note that $P_{Busy} + P_{Br} = \lambda\,E(X)\,E(H_c) = \rho_c$

iv. P_D = the probability that the server is doing his setup work
= $E(D)\,P_1(0)$

Mean system size

Let L_N denote the expected system size of the unreliable $M^{[X]}/G/1$ queue with N-policy, single vacation and setup time.

Then $L_N = \left(\dfrac{d}{dz} P(z)\right)_{z=1}$.

By calculation, $L_N = \dfrac{(\lambda E(X))^2 E(H_c^2) + \lambda E(X(X-1))E(H_c)}{2(1-\rho_c)} + \rho_c$

$$+ \frac{\left(\lambda E(X)E(D)\sum_{n=0}^{N-1}\psi_n + \sum_{n=0}^{N-1} n\,\psi_n\right) + \dfrac{\lambda^2 E(X)}{2}\,(E(D^2) + 2E(D)E(V) + E(V^2))}{\lambda(E(D)+E(V)) + \sum_{n=0}^{N-1}\psi_n}$$

Expected Cycle Length

Let $E(T_N)$, $E(B)$, $E(T_c)$, $E(Br)$, $E(D)$ and $E(C)$ represent the expected idle period, expected busy period, expected cycle, expected break down period, expected setup period and the expected completion period respectively.Then the long-run fraction of time the server is idle and busy are given by,

$$\frac{E(T_N)}{E(T_{cy})} = P_I = E(V)\,P_1(0)$$

$$\frac{E(B)}{E(T_{cy})} = P_{Busy} = \rho_{Busy}$$

$$\frac{E(Br)}{E(T_{cy})} = P_{Br} = \rho_{Br}$$

$$\frac{E(D)}{E(T_{cy})} = P_D = E(D)\, P_1(0)$$

From the above calculations $E(T_{cy}) = \dfrac{1}{P_1(0)} = \dfrac{E(D) + E(V) + \sum_{n=0}^{N-1} \frac{\psi_n}{\lambda}}{1-\rho_c}$

Then E(C) = Expected completion period

$$= E(B) + E(Br)$$

$$= (\rho_{Busy} + \rho_{Br})\, E(T_{cy})$$

$$= \frac{\rho_c}{1-\rho_c}\left(E(D) + E(V) + \sum_{n=0}^{N-1} \frac{\psi_n}{\lambda}\right)$$

Optimal Design of N-policy with Single Vacation

In this section, the total expected cost function per unit time for the model to calculate the optimal value N_S^* which minimizes the linear cost function is developed. A cost structure that has been widely used in literature is employed.

Let

C_y - turn on cost per cycle
C_h - holding cost per unit time
C_D - setup cost per unit time
C_V - reward per unit time due to vacation
C_{Busy} - operating cost per unit time
C_{Br} - breakdown cost per unit time
$T_c(N)$ - total average cost per unit time
C_I - idle cost per unit t

Then $T_c(N) = \dfrac{[C_y + C_D\, E(D) + C_I\, E(T_N) - C_V\, E(V)]}{E(T_{cy})} + C_h\, L_N + C_{Busy}\, P_{Busy} + C_{Br}\, P_{Br}$

$$= \frac{1-\rho_c}{[E(D)+E(V)] + \sum_{n=0}^{N-1}\frac{\psi_n}{\lambda}}\left[C_y + C_D\, E(D) + C_I \sum_{n=0}^{N-1}\frac{\psi_n}{\lambda} - C_V\, E(V)\right] +$$

$$C_h\left(L_1 + \frac{\left(\lambda\, E(X)\, E(D)\sum_{n=0}^{N-1}\psi_n + \sum_{n=0}^{N-1} n\,\psi_n\right) + \frac{\lambda^2 E(X)}{2}\left(E(D^2) + 2E(D)\,E(V) + E(V^2)\right)}{\lambda\,(E(D)+E(V)) + \sum_{n=0}^{N-1}\psi_n}\right) + C_{Busy}\,\rho_{Busy} + C_{Br}\,\rho_{Br}$$

Then $T_c(N) = \dfrac{A + (\lambda((1-\rho_c)C_I + \lambda E(X)E(D)C_h))\sum_{n=0}^{N-1}\dfrac{\psi_n}{\lambda} + C_h\sum_{n=0}^{N-1} n\,\psi_n}{C_N} + A_1$

where $A_1 = C_{Busy}\,\rho_{Busy} + C_{Br}\,\rho_{Br} + C_h\,L_1$

$A = \lambda(1-\rho_c)(C_y + C_D E(D) - C_V E(V)) + C_h \dfrac{\lambda^2 E(X)}{2}(E(D^2) + 2E(D)E(V) + E(V^2))$

$C_N = \lambda(E(D) + E(V)) + \sum_{n=0}^{N-1}\psi_n$

and $L_1 = \rho_c + \dfrac{(\lambda E(X))^2 E(H_c^2) + \lambda E(X(X-1))E(H_c)}{2(1-\rho_c)}$

Theorem

Let N_S^* be the optimal threshold value of N that minimizes the average cost per unit time $T_c(N)$ under the cost structure mentioned. Then N_S^* of the model is given by,

$$N_S^* = \min\left\{ k \geq 1 / (\lambda(E(D)+E(V)))(C_{Busy}(1-\rho_c) + \lambda E(X)E(D)\;C_h + k\,C_h) + C_h\sum_{n=0}^{k}(k-n)\,\psi_n) > A \right\}$$

Proof

By calculation, $T_c^S(k+1) - T_c^S(k) = \dfrac{\psi_k}{C_k\,C_{k+1}}\,(h(k))$,

where $h(k) = -A + \lambda(E(D) + E(V))(C_{Busy}(1-\rho_c) + \lambda E(X)E(D)C_h + k\,C_h) + C_h\sum_{n=0}^{k}(k-n)\,\psi_n$

The sign of h(k) determines whether $T_c(k)$ increases (or) decreases.

If k be the first integer such that h(k) > 0, then

$h(k+1) = h(k) + C_h(\lambda(E(D) + E(V))) + C_h\,\Psi_n$
> 0

This implies h(k + 1) > 0 whenever h(k) > 0

Therefore N_S^* = first k,for which h(k) > 0

(i.e.) $N_S^* = \min\{k \geq 1/h(k) > 0\}$

NUMERICAL ANALYSIS

Numerical results are provided for,

(i) expected number of customers waiting in the system (L_N)
(ii) the optimal value N^* of N
(iii) the total expected cost per unit time ($T_c(N)$)
(iv) expected busy period E(B)
(v) expected length of the cycle $E(T_{cy})$ for the model.

For the computation work of the model, we make the following assumptions:

- The batch size X follows the geometric distributions (i.e.) $g_k = \Pr(X = k) = (1 - p)\, p^{k-1}$, $k \geq 1$, with mean $E(X) = \frac{1}{1-p}$
- The service time S of each stage follows two-stage hyper exponential distributions whose measures are the following :

  The mean of S_i, i = 1, 2 are $E(S_1) = \frac{a_1}{\mu_{11}} + \frac{a_2}{\mu_{12}}$ and $E(S_2) = \frac{b_1}{\mu_{21}} + \frac{b_2}{\mu_{22}}$

  The second order moment of S_i, i = 1, 2 are

  $E(S_1^2) = 2\left(\frac{a_1}{\mu_{11}^2} + \frac{a_2}{\mu_{12}^2}\right)$ and $E(S_2^2) = 2\left(\frac{b_1}{\mu_{21}^2} + \frac{b_2}{\mu_{22}^2}\right)$
- The set-up time D and vacation V follow Erlang 3-type distributions with mean $E(D) = \frac{1}{\gamma}$, $E(V) = \frac{1}{\eta}$ and the second order moments $E(D^2) = \frac{4}{3\gamma^2}$ and $E(V^2) = \frac{4}{3\eta^2}$.
- The repair time follows exponential distribution with parameters β_i, i = 1, 2

$C_h = 10$, $C_D = 250$, $C_y = 200$, $C_V = 1$, $C_{Busy} = 100$, $C_{Br} = 5$.

The parameters a_i, b_i, i = 1, 2 are the same for all the tables. $a_1 = 0.1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 3$.

In Table (1), it is verified that as the arrival rate λ increases, the optimal expected system size and the total expected cost also increase. The expected length of the cycle and the corresponding utilization factor ρ and ρ_h are also given in the table for particular values of p, γ and η.

Table – 1 **$p = 0.01, \gamma = 0.3, \eta = 0.1$**

	Ns^*	$E(T_{Cy})$	L_N	$T_C(N)$	ρ	ρ_h
0.25	7	0.0214	4.7058	92.3929	0.2062	0.4124
0.31	5	0.0291	5.2337	105.6724	0.2557	0.5114
0.37	4	0.0143	6.9628	126.3522	0.3052	0.6104
0.45	4	0.0132	11.7416	173.7736	0.3712	0.7424
0.5	4	0.0105	18.1998	237.4419	0.4124	0.8249

In queueing models, it is obvious that, if the mean service time is reduced, the expected queue size will be reduced. $E(S) = E(S_1) + r * E(S_2) = \left(\frac{a_1}{\mu_{11}} + \frac{a_2}{\mu_{12}}\right) + r\left(\frac{b_1}{\mu_{21}} + \frac{b_2}{\mu_{22}}\right)$ can be reduced by increasing any one of the μ_{ij} values. Then to justify the statement, that queue length is an increasing function of the mean service time E(S), for increasing values of μ_{ij} the measures are calculated. The optimal total cost of the system and the expected busy period are also presented in each table. The value of r is fixed in tables (2) to (5) as r = 0.3.

E(S2) = 7, p = 0.01

Table – 2 **$\mu_{11} = 1.5, \mu_{21} = 2, \mu_{22} = 0.5$**

μ_{12}	E(S)	N_S^*	E(B)	L_N	$T_C(N)$	ρ	ρ_h
0.81	4.0184	1	114.4432	28.2754	338.1308	0.4844	0.9689
0.83	3.9738	1	64.7710	16.8881	226.3452	0.4732	0.9464
0.85	3.9313	1	45.1903	12.3868	183.3204	0.4624	0.9249
1	3.6666	3	28.9075	5.9391	116.3661	0.3956	0.7912
1.3	3.3204	4	15.4381	4.4027	99.5371	0.3082	0.6164

E(S1) = 0.8166

Table – 3 **$\mu_{11} = 1.5, \mu_{12} = 2, \mu_{22} = 0.5$**

μ_{21}	E(S)	N_S^*	E(B)	L_N	$T_C(N)$	ρ	ρ_h
0.5	5.8	6	35.9684	12.1108	112.4507	0.4853	0.9807
1	3.2166	6	28.5709	10.8934	93.0612	0.4671	0.9489
1.5	3.0165	4	21.3217	8.6034	82.8101	0.3782	0.8091
2	2.9166	5	14.4494	6.7809	72.8520	0.3391	0.7925
2.5	2.8566	5	9.2149	5.3628	67.1350	0.2034	0.4068

It is shown in Tables (4) and (5) that the expected queue length can be decreased by reducing the setup time E(D) and reducing the vacation time E(V). The fixed values of $\lambda = 0.25$ and $p = 0.2$ are considered to justify the result.

Table – 4 **$\eta = 0.1$**

γ	E(D)	N_S^*	L_N	$T_C(N)$
0.01	100	2	24.7269	384.3603
0.03	33	7	12.0618	223.6921
0.05	20	7	9.3591	182.0197
0.1	10	6	7.0617	142.8269
0.3	3.3333	5	4.0055	89.7542

Table – 5 **$\gamma = 0.3$**

η	E(V)	N_S^*	L_N	$T_C(N)$
0.01	100	17	18.4688	211.6438
0.02	50	9	10.0110	132.3620
0.03	33.3333	7	7.2221	108.9538
0.05	20	5	5.0678	94.4087
0.1	10	4	3.7140	89.8612

The values of Table(6) give the optimum value N^* of N and the minimum optimal expected cost for the N-policy queueing model corresponding to different cost structure.

Table – 6
$(\gamma, a_1, a_2, b_1, b_2, \mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \eta) = (0.3, 0.1, 1.5, 2, 3, 1.5, 2.0, 6, 0.5, 0.1)$

Case : 1 **$C_h = 5, C_D = 250, C_{Busy} = 100$**

(λ, p)	(.1, .2)	(0.15, 0.23)	(.2, .26)	(.25, .29)	(.3, .32)
N_S^*	9	10	10	9	7
$T_C(N)$	66.8890	82.7920	97.3215	112.7271	137.0123

Case : 2 **$C_h = 15, C_D = 500, C_{Busy} = 150$**

(λ, p)	(.1, .2)	(.15, .23)	(.2, .26)	(.25, .29)	(.3, .32)
N_S^*	7	8	7	7	5
$T_C(N)$	143.1425	179.2090	212.715	251.6618	321.6875

Case : 3 **$C_h = 25, C_D = 750, C_{Busy} = 200$**

(λ, p)	(.1, .2)	(.15, .23)	(.2, .26)	(.25, .29)	(.3, .32)
N_S^*	6	7	7	6	5
$T_C(N)$	218.5567	273.9759	326.5689	389.6246	505.7298

Case : 4 **C_h = 35, C_D = 1000, C_{Busy} = 250**

(λ, p)	(.1, .2)	(.15, .23)	(.2, .26)	(.25, .29)	(.3, .32)
N_S^*	6	7	7	6	5
$T_C\,(N)$	293.2836	368.7054	440.4227	527.3215	689.7720

The following table values justify the procedure (given in the theory) of finding the optimal values N^* of N for this model. The parametric values assumed to obtain the optimal values N^* are mentioned in the following table.

Table – 7

p = 0.03, a_1 = 1.5, a_2 = 2, b_1 = 2.5, b_2 = 3.5, μ_{11} = 2, μ_{12} = 2.5, μ_{21} = 3, μ_{22} = 4, λ= 0.15, C_h = 5,

C_D = 4000, C_{Busy} = 100, C_y = 250, C_V = 5, C_{Br} = 8

	N	L_N	$T_C\,(N)$
	3	3.7082	67.3904
	4	4.1582	65.2486
N^*	**5**	**4.6283**	**64.8336**
	6	5.1080	65.3364
	7	5.5932	66.3825

Conclusion

This is an extension of the work on Non- Markovian queueing system combining N - Policy with setup time and vacation,carried out by several researchers including Medhi and Templeton [5], Minh [6], Lee and Park [4], Lee et al. [7], Hur and Paik [8]. But these authors have focused only on reliable servers. In this paper, for the Non- Markovian unreliable queueing system with N- Policy,second optional service,setup time and vacation,the PGF of the system size is presented in closed form.Further, various performance measures are derived and the numerical analysis is carred out to verify that the mean system size is an increasing function of arrival rate λ, decreasing function of service rate μ, setup rate γ and vacation parameter η. Further the optimal value of N that minimizes the total cost is calculated numerically for various parametric values and the procedures given above are justified.

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