American. Jr. of Mathematics and Sciences
Vol. 1, No.1 ,(January 2012)

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Study of an Exact Solution of Unsteady-State Thermoelastic Problem of a Circular Plate

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Abstract

The aim of this work is to determine the unknown temperature, displacementand thermal stresses on the upper surface of a circular plate subjected to an interior heat flux is known under unsteady-state field. The lower surface is kept at zero temperature and the fixed circular edge is thermally insulated. The governing heat conduction equation has been solved by using the Hankel and Laplace transform technique. The results are obtained in series form in terms of Bessel's functions and results have been computed numerically and illustrated graphically.

1. Introduction

As known, thermal behaviors of structures must be considered in many situations. Study of thermal effect on deformations and stresses of a plate, especially a circular plate is increasingly important. Firstly, the problems of circular plates are more complicated and thus more attractive to many scientists. Secondly, there are practical requirements for thick plates in various modern projects, such as high building, raceway, high-way, container wharf, and so on.

Ashida et al. [1] discussed the inverse transient thermoelastic problem for a composite circular plate. Tikhe et al. [2] solved an inverse heat conduction problem in a thin circular plate and its thermal deflection. Deshmukh et al. [3] are also discussed on an inverse transient problem of quasi-static thermal deflection of a thin clamped circular plate. Grysa et al [4] studied the one dimensional problem of temperature and the heat flux at the surface of a thermo elastic slab. Kulkarni et al. [5] studied an inverse transient problem of quasi-static thermal stresses in a thick circular plate. Also Roy Choudhary [7] studied a rate of quasi-static stress in a thin circular plate due to transient temperature applied along the circumference of a circle over the upper face.

Here an attempt is made to solve an inverse unsteady-state thermoelastic problem in a circular plate to determine the unknown temperature, displacement and stress components on the upper surface $(z = h)$ of a circular plate subjected to an interior flux $f(r, t)$ is known under unsteady-state field. The lower surface $(z = -h)$ is kept at zero temperature and the fixed circular edge $(r = a)$ is thermally insulated. The governing heat conduction equation has been solved by using the Hankel and Laplace transform technique. The results are obtained in series form in terms of Bessel's functions and results have been computed numerically and illustrated graphically.

This paper contains new and novel contribution of stresses in circular plate under steady state. The results presented here will be more useful in engineering problem

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**2000 Mathematics Subject classifications*: Primary 35A25, secondary 74M99, 74K20.

Key words and phrases: Unsteady State, Thermoelastic problem, Hankel transform, Circular Plate.

particularly in the determination of the state of strain in circular plate constituting foundations of containers for hot gases or liquids, in the foundations for furnaces etc.

2. Statement of the problem

Consider a circular plate of radius a and thickness $2h$ occupying space $\mathrm{D} : 0 \le r \le a$, $0 \le z \le h$. Initially the plate is at zero temperature. Let the plate be subjected to an interior heat flux $f(r,t)$ is known within region $-h \le z \le h$. The lower surface $(z = -h)$ is kept at zero temperature and the fixed circular edge $(r = a)$ is thermally insulated. Assume that the boundary of the circular plate is free from traction. Under these more realistic prescribed conditions, the unknown temperature $g(r,t)$ which is at the upper surface of the plate and the thermal stresses due to unknown temperature $g(r, t)$ need to be determined.

The differential equation governing the displacement potential function is given in [6] as,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \mathrm{K}\tau \tag{1}$$

where K is the restrain coefficient and the temperature change is given by $\tau = T - T_i$ where T_i is the initial temperature. The displacement function ϕ is known as Goodier's thermoelastic potential.

The unsteady state temperature of the plate at time t satisfies the heat condition equation is,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2}$$

where k is the thermal diffusivity of the material of the plate.

The initial and boundary conditions are

$$T(r, z, t) = 0 \quad \text{at } t = 0, \tag{3}$$

The boundary condition

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = a, -h \le z \le h \tag{4}$$

$$\frac{\partial T}{\partial z} = 0 \quad \text{at } \mathrm{z} = \text{-h}, 0 \le r \le a \tag{5}$$

$$\frac{\partial T}{\partial z} = g(r, t) \; (unknown) \quad \text{at } \mathrm{z} = h, 0 \le r \le a \tag{6}$$

$$\frac{\partial T}{\partial z} = f(r, t) \; (known) \quad \text{at } \mathrm{z} = \xi, 0 \le r \le a, -h \le \xi \le h \tag{7}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by,

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial U}{\partial r} \tag{8}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \tag{9}$$

where μ is the Lame's constant, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the plate in the plane state of stress. The equations (1) to (9) constitute the mathematical formulation of the problem under consideration.

3. Solution of the Problem

Applying finite Hankel transform defined in [8] to the Eq. (2) to Eq. (7), one obtain

$$\frac{d^2 \overline{T}}{dz^2} - \lambda_n^2 \overline{T} = \frac{1}{k}\frac{d\overline{T}}{dt} \tag{10}$$

with boundary conditions

$$\frac{\partial \overline{T}}{\partial r} = 0 \quad \text{at } r = a,\ -h \le z \le h \tag{11}$$

$$\frac{\partial \overline{T}}{\partial z} = 0 \quad \text{at } z = -h,\ 0 \le r \le a \tag{12}$$

$$\frac{\partial \overline{T}}{\partial z} = \overline{g}(\lambda_n, t) \quad \text{at } z = h,\ 0 \le r \le a \tag{13}$$

$$\frac{\partial \overline{T}}{\partial z} = \overline{f}(\lambda_n, t) \quad \text{at } z = \xi,\ 0\text{-}h \le \xi \le h \tag{14}$$

where $\overline{T}$ denotes the Hankel transform of T and λ_n is the Hankel transform parameter. Again applying Laplace transform defined in [8] to the Eq. (10) to Eq. (14) .we get,

$$\frac{d^2 \overline{T}^*}{dz^2} - q^2 \overline{T}^* = 0 \tag{15}$$

where

$$q^2 = \left(\lambda_n^2 + \frac{s}{k}\right)$$

$$\frac{d\overline{T}^*}{dz}(\lambda_n, \xi, s) = \overline{f}^*(\lambda_n, s)$$

at $z = \xi,\ 0 \le r \le a$ (16)

$$\frac{d\overline{T}^*}{dz}(\lambda_n, h, s) = \overline{g}^*(\lambda_n, s) \quad \text{at } z = h,\ 0 \le r \le a \tag{17}$$

where $\overline{T}^*$ denotes the Laplace transform of $\overline{T}$ and s is the Laplace transform parameter.

The Eq. (15) is a second order differential equation, whose solution is given by,

$$\overline{T}^*(\lambda_n, z, s) = A e^{qz} + B e^{-qz} \tag{18}$$

where A and B are constants.

Using Eq.(16) and Eq.(17) in Eq.(18) we obtain the values of A and B. Substituting these values in Eq.(18) and then inversion of finite Laplace and finite Hankel integral transform leads to,

$$T(r,z,t) = \frac{2k\pi}{h^2} \sum_{n=1}^{\infty} \frac{J_0(r\lambda_n)}{\lambda_n [J_0(a\lambda_n)]^2} \sum_{m=1}^{\infty} (-1)^{(m+1)}\, m \cos\left(\frac{m\pi(z-h)}{(\xi-h)}\right)$$

$$\int_0^t \frac{\overline{f}(\lambda_n, t')}{\lambda_n}\, e^{-k\left[\lambda_n^2 + \frac{m^2\pi^2}{h^2}\right](t-t')}\, dt'$$

$$-\frac{2k\pi}{h^2}\sum_{n=1}^{\infty}\frac{J_0(r\lambda_n)}{\lambda_n[J_0(a\lambda_n)]^2}\sum_{m=1}^{\infty}(-1)^{(m+1)}\, m\cos\left(\frac{m\pi(z-\xi)}{(\xi-h)}\right)$$

$$\int_0^t \frac{\bar{g}(\lambda_n,t')}{\lambda_n}\, e^{-k\left[\lambda_n^2+\frac{m^2\pi^2}{h^2}\right](t-t')}\, dt' \qquad (19)$$

since $T_i = 0$, the temperature change is $\tau = T - T_i = T$.

3. Determination of Thermoelastic Displacement

On putting the values of temperature $T(r,z,t)$ from Eq. (19) in Eq. (1), one obtain the thermoelastic displacement function $\phi(r, z, t)$ as,

$$\phi(r,z,t) = \mathrm{K}\frac{k\pi}{2h^2}\sum_{n=1}^{\infty}\frac{r^2 J_0(r\lambda_n)}{\lambda_n[J_0(a\lambda_n)]^2}\sum_{m=1}^{\infty}(-1)^{(m+1)}\, m\cos\left(\frac{m\pi(z-h)}{(\xi-h)}\right)$$

$$\times\int_0^t \frac{\bar{f}(\lambda_n,t')}{\lambda_n}\, e^{-k\left[\lambda_n^2+\frac{m^2\pi^2}{h^2}\right](t-t')}\, dt'$$

$$-\mathrm{K}\frac{k\pi}{2h^2}\sum_{n=1}^{\infty}\frac{r^2 J_0(r\lambda_n)}{\lambda_n[J_0(a\lambda_n)]^2}\sum_{m=1}^{\infty}(-1)^{(m+1)}\, m\cos\left(\frac{m\pi(z-\xi)}{(\xi-h)}\right)$$

$$\times\int_0^t \frac{\bar{g}(\lambda_n,t')}{\lambda_n}\, e^{-k\left[\lambda_n^2+\frac{m^2\pi^2}{h^2}\right](t-t')}\, dt' \qquad (20)$$

The radial displacement U as,

$$U(r,z,t) = \mathrm{K}\frac{k\pi}{2h^2}\sum_{n=1}^{\infty}\frac{\left[2rJ_0(r\lambda_n)-r^2J_1(r\lambda_n)\right]}{\lambda_n[J_0(a\lambda_n)]^2}\sum_{m=1}^{\infty}(-1)^{(m+1)}\, m\cos\left(\frac{m\pi(z-h)}{(\xi-h)}\right)$$

$$\times\int_0^t \frac{\bar{f}(\lambda_n,t')}{\lambda_n}\, e^{-k\left[\lambda_n^2+\frac{m^2\pi^2}{h^2}\right](t-t')}\, dt'$$

$$-\mathrm{K}\frac{k\pi}{2h^2}\sum_{n=1}^{\infty}\frac{\left[2rJ_0(r\lambda_n)-r^2J_1(r\lambda_n)\right]}{\lambda_n[J_0(a\lambda_n)]^2}\sum_{m=1}^{\infty}(-1)^{(m+1)}\, m\cos\left(\frac{m\pi(z-\xi)}{(\xi-h)}\right)$$

$$\times \int_0^t \frac{\bar{g}(\lambda_n, t')}{\lambda_n} e^{-k\left[\lambda_n^2 + \frac{m^2\pi^2}{h^2}\right](t-t')} dt'. \tag{21}$$

4. Determination of Stress functions

Using Eq. (21) in Eq. (8) and Eq. (9), the stress functions are obtained as,

$$\sigma_{rr} = 2\mathrm{K}\frac{\mu k\pi}{h^2}\sum_{n=1}^{\infty}\frac{\left[J_1(r\lambda_n) - \frac{J_0(r\lambda_n)}{r} + rJ_1'(r\lambda_n) + J_1(r\lambda_n)\right]}{\lambda_n[J_0(a\lambda_n)]^2}$$

$$\times \sum_{m=1}^{\infty}(-1)^{(m+1)} m\cos\left(\frac{m\pi(z-h)}{(\xi-h)}\right)\int_0^t \frac{\bar{f}(\lambda_n, t')}{\lambda_n} e^{-k\left[\lambda_n^2 + \frac{m^2\pi^2}{h^2}\right](t-t')} dt'$$

$$-2\mathrm{K}\frac{\mu k\pi}{h^2}\sum_{n=1}^{\infty}\frac{\left[J_1(r\lambda_n) - \frac{J_0(r\lambda_n)}{r} + rJ_1'(r\lambda_n) + J_1(r\lambda_n)\right]}{\lambda_n[J_0(a\lambda_n)]^2}$$

$$\times \sum_{m=1}^{\infty}(-1)^{(m+1)} m\cos\left(\frac{m\pi(z-\xi)}{(\xi-h)}\right)\int_0^t \frac{\bar{f}(\lambda_n, t')}{\lambda_n} e^{-k\left[\lambda_n^2 + \frac{m^2\pi^2}{h^2}\right](t-t')} dt' \tag{22}$$

$$\sigma_{\theta\theta} = 2\mathrm{K}\frac{\mu k\pi}{h^2}\sum_{n=1}^{\infty}\frac{\left[4rJ_1'(r\lambda_n) + r^2J_1''(r\lambda_n) + 3J_1(r\lambda_n)\right]}{\lambda_n[J_0(a\lambda_n)]^2}$$

$$\times \sum_{m=1}^{\infty}(-1)^{(m+1)} m\cos\left(\frac{m\pi(z-h)}{(\xi-h)}\right)\int_0^t \frac{\bar{f}(\lambda_n, t')}{\lambda_n} e^{-k\left[\lambda_n^2 + \frac{m^2\pi^2}{h^2}\right](t-t')} dt'$$

$$-2\mathrm{K}\frac{\mu k\pi}{h^2}\sum_{n=1}^{\infty}\frac{\left[4rJ_1'(r\lambda_n) + r^2J_1''(r\lambda_n) + 3J_1(r\lambda_n)\right]}{\lambda_n[J_0(a\lambda_n)]^2}$$

$$\times \sum_{m=1}^{\infty}(-1)^{(m+1)} m\cos\left(\frac{m\pi(z-\xi)}{(\xi-h)}\right)\int_0^t \frac{\bar{g}(\lambda_n, t')}{\lambda_n} e^{-k\left[\lambda_n^2 + \frac{m^2\pi^2}{h^2}\right](t-t')} dt'. \tag{23}$$

5. Numerical Calculation

Numerical calculations have been carried out for a steel (SN 50C) plate with parameters chosen $a = 1m,\ h = 1m,\ \xi = \frac{h}{2}$, $t = 10\,\text{sec}$. The thermal diffusivity is given by

$k = 15.9\times 10^{6}\,(m^{2}s^{-1})$. The Poisson ratio by $\nu = 0.281$ and $\lambda = 59.0\,Wm^{-1}K^{-1}$.

The transcendental roots of $J_1(\lambda_n a)$ as in [9] are $J_1 = 3.8317$, $J_2 = 7.0156$, $J_3 = 10.1735$,

$J_4 = 13.3237,\ J_5 = 16.470,\ J_6 = 19.6159$.

For convenience, we get

$A_n = \dfrac{2k\pi}{\lambda_n[J_0\,(a\lambda_n)]^2 h^2}$, $B_n = \mathrm{K}\,\dfrac{k\pi}{2\lambda_n[J_0\,(a\lambda_n)]^2 h^2}$ and $C_n = 4\mu\,\mathrm{K}B_n$

These values are used to evaluate the temperature, displacement, stress components and thermal stresses given by Eqs.21-25 these have been computed numerically and illustrated graphically with help of a computer programme.

6. Concluding Remarks

In this article, we study an inverse unsteady-state thermal stresses in a circular plate. We develop the analysis for the temperature field by introducing the methods of the Hankel transforms and Laplace transforms and determine the unknown temperature, displacement, stress components on the upper surface.

From fig.1 and fig.2, we observe that temperature decreases from lower surface to outer circular surface in radial direction. Also it decreases from upper surface to lower surface in radial direction. From fig.3 & fig.4 thermoelastic displacement decreases from inner circular surface to outer circular surface in axial direction. Also it increases from lower surface to upper surface. From fig.5 & fig.6 stress function develops the tensile stresses in radial direction and also decreases to lower surface. From fig.7 & fig.8 stress function decreases from inner circular surface to outer circular surface in axial direction and it develops the tensile stresses in radial direction.

American. Jr. of Mathematics and Sciences
Vol. 1, No.1 ,(January 2012)

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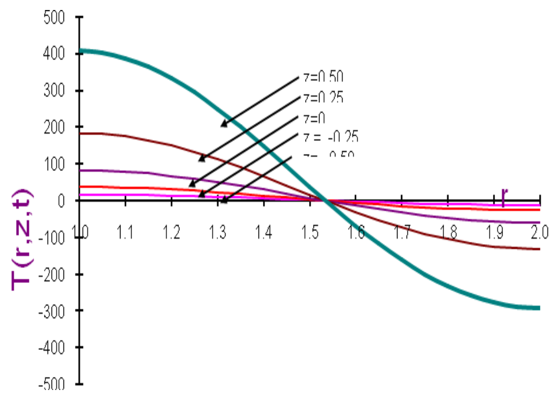

Fig.1 The temperature distribution T(r,z,t) In radial direction

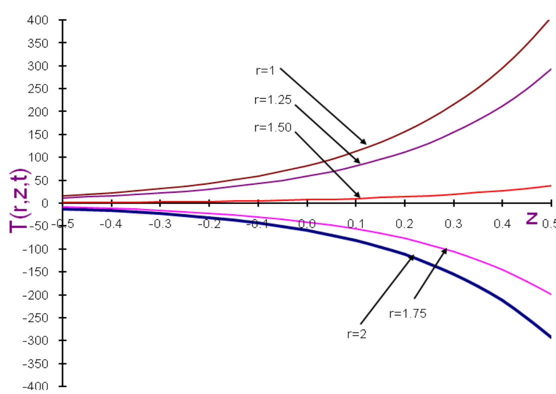

Fig.2 The temperature distribution T(r,z,t) in axial direction

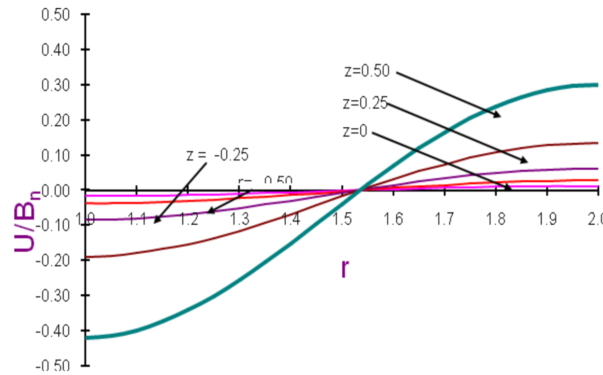

Fig.3 The displacement function U(r,z,t) in radial direction

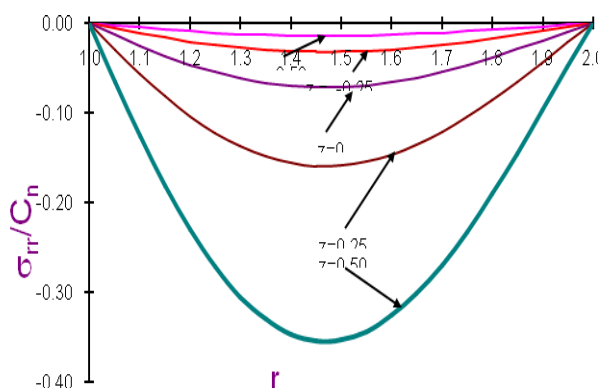

Fig.5 The stress function σ_{rr} in radial direction

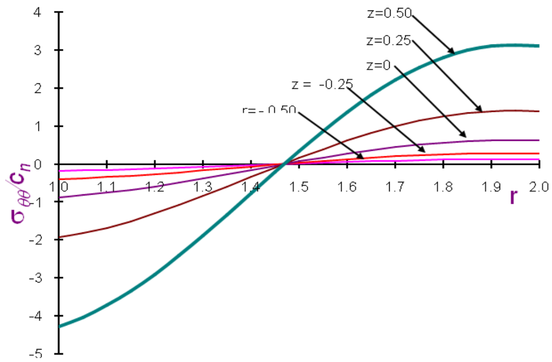

Fig.7. The stress function $\sigma_{\theta\theta}$ in radial direction

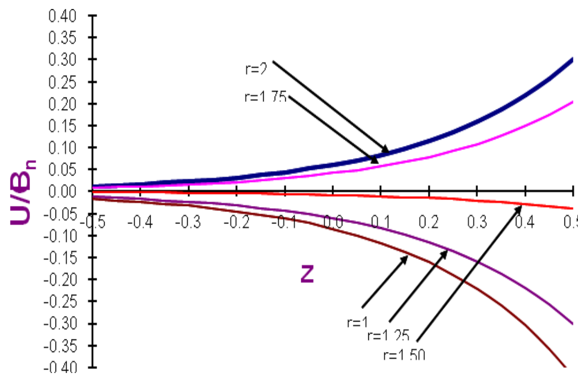

Fig.4 The displacement function U(r,z,t) in axial direction

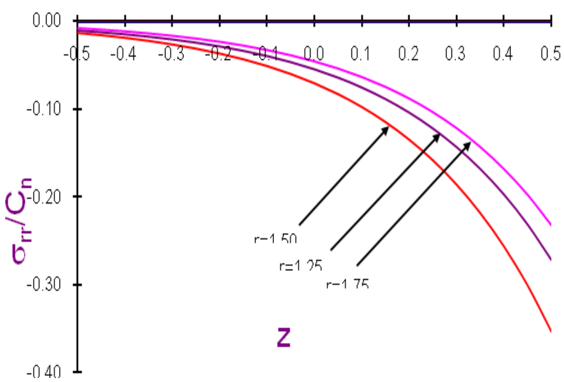

Fig.5 The stress function σ_{rr} in axial direction

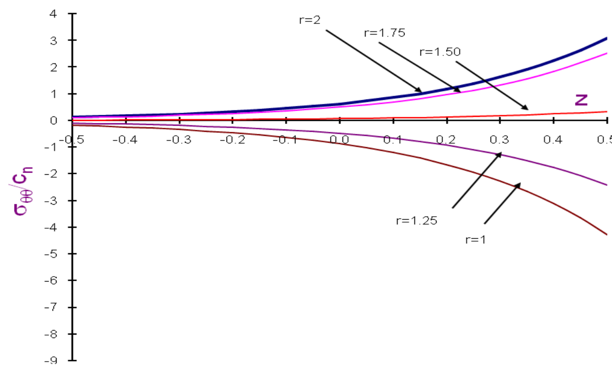

Fig.8. The stress function $\sigma_{\theta\theta}$ in axial direction

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