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Sufficient Conditions for Univalence of an Integral Operator Defined by Generalized Differential Operator

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Abstract

In this paper, we investigate the univalence of an integral operator defined by generalized differential operator.

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1. Introduction

Let A denote the class of functions $f(z)$ of the form

(1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk U : $\{z \in C : |z| < 1\}$, and $S := \{f \in A : f$ is univalent in U$\}$.

For functions f given by (1.1) and g given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ ($q, s \in N \cup \{0\}, q \le s + 1$) be complex numbers such that $\beta_k \neq 0, -1, -2, \ldots$ for $k \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \frac{z^n}{n!}$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = x(x+1) \ldots (x+n-1) \text{ for } n \in N \text{ and } (x)_0 = 1.$$

Corresponding to a function $\mathcal{G}_{q,s}(\alpha_1, \beta_1; z)$ defined by

$$\mathcal{G}_{q,s}(\alpha_1, \beta_1; z) = z\,{}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),$$

For $f \in A$, C. Selvaraj [1] introduced the following generalized differential operator:

(1.2) $$D^0_{\lambda\mu}(\alpha_1, \beta_1) f(z) = f(z) * \mathcal{G}_{q,s}(\alpha_1, \beta_1; z),$$

(1.3) $$D^1_{\lambda\mu}(\alpha_1,\beta_1)f(z) = D_{\lambda\mu}(\alpha_1,\beta_1)f(z)$$
$$= \lambda\mu z^2(f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z))'' + (\lambda-\mu)z(f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z))'$$
$$+(1-\lambda-\mu)(f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z)),$$

and

(1.4) $$D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z) = D_{\lambda\mu}(D^{m-1}_{\lambda\mu}(\alpha_1,\beta_1)f(z))$$

where $0 \le \mu \le \lambda \le 1$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1), then from (1.3) and (1.4) we see that

(1.5) $$D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z) = z + \sum_{n=2}^{\infty} \theta_n^m \sigma_n a_n z^n$$

where

(1.6) $$\theta_n = [1 + (\lambda\mu\, n + \lambda - \mu)(n-1)]$$

and

$$\sigma_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_q)_{n-1}}\frac{1}{(n-1)!}.$$

It can be seen that, by specializing the parameters the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to many known and new differential operators. In particular, when m = 0, the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to the well known Dziok-Srivastava operator and for $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, it reduces to the operator introduced by F.Al. Oboudi. Further we remark that when $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to the operator introduced by G.S. Sălăgean.

For simplicity, in the sequel, we will write $D^m_{\lambda\mu}f(z)$ instead of $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$.

Definition 1. Let $m, n \in \mathbb{N}_0$ and $\delta_i \in \mathbb{C}$, $1 \le i \le n$, we define the integral operator
$I_{\lambda\mu}(f_1,\ldots,f_n) = \mathcal{A}^n \to \mathcal{A}$,

$$I_{\lambda\mu}(f_1,\ldots,f_n)(z) := \left([1+(\lambda\mu n+\lambda-\mu)(n-1)]\int_0^z \left(\frac{D^m_{\lambda\mu}f_1(t)}{t}\right)^{\delta_1} \cdots \left(\frac{D^m_{\lambda\mu}f_n(t)}{t}\right)^{\delta_n} dt \right)^{1+(\lambda\mu n+\lambda-\mu)(n-1)}, \quad (z \in \mathbb{U})$$

where $f_i \in A$ and $D^m_{\lambda\mu}$ is the generalized differential operator.

Remark 1. (i) For $m = 0$, $n = 1$, $\delta_1 = 1$, $\delta_2 = \delta_3 = \cdots = \delta_n = 0$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, $D^0_0 f_1(z) := D^0 f(z) = f(z) \in \mathcal{A}$, we have the Alexander integral operator

$$I(f)(z) := \int_0^z \frac{f(t)}{t}dt$$

was introduced [2].

(ii) For $m = 0$, $n = 1$, $\delta_1 = \delta \in [0, 1]$, $\delta_2 = \delta_3 = \cdots = \delta_n = 0$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, and $D^0_0 f_1(z) = D^0 f(z) = f(z) \in S$, we have the integral operator

$$I_\delta(f)(z) := \int_0^z \left(\frac{f(t)}{t}\right)^\delta dt$$

was studied in [3].

(iii) For $m = 0$, $n \in N_0$, $\delta_i \in C$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, $D_0^0 f_i(z) = f_i(z) \in S$, $1 \le i \le n$, we have the integral operator

$$I(f_1,\ldots,f_n)f(z) := \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\delta_i} dt$$

was studied in [4].

(iv) For $m = 0$, $n = 1$, $\delta_1 = \gamma$, $\delta_2 = \delta_3 = \cdots = \delta_n = 0$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, $D_0^0 f_1(z) = D^0 f(z) = f(z) \in S$, we have the integral operator

$$I_\gamma(f)(z) := \int_0^z \left[\frac{f(t)}{t}\right]^\gamma dt,$$

was studied in [5] and [6].

2. Main Results

The following lemmas will be required in our investigation.

Lemma 2.1 (see [7]) If the function f is regular in the unit disk U, $f(z) = z + a_2 z^2 + \cdots$ and

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1$$

for all $z \in$ U, then the function f is univalent in U.

Lemma 2.2 (Schwarz Lemma) (see [8, p. 166]) Let the analytic function $f(z)$ be regular in U and let $f(0) = 0$. If, z in U, $|f(z)| \le 1$, then

$$|f(z)| \le |z|, \quad (z \in U)$$

and $|f'(0)| \le 1$. The equality holds if and only if $f(z) \equiv kz$ and $|k| = 1$.

Theorem 2.3 Let $m, n \in N_0$, $\delta_i \in C$ and $f_i \in A$; $1 \le i \le n$. If

$$\left|\frac{z\left(D_{\lambda\mu}^m f_i(z)\right)'}{D_{\lambda\mu}^m f_i(z)} - 1\right| \le 1$$

and $|\delta_1| + \cdots + |\delta_n| \le 1$, then $I_{\lambda\mu}(f_1, \ldots, f_n)(z)$ defined in Definition 1 is univalent in U.

Proof. Since $f_i \in A$, $1 \le i \le n$, by (1.5), we have

$$\frac{D_{\lambda\mu}^m f_i(z)}{z} = 1 + \sum_{n=2}^\infty [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m \frac{(\alpha_1)_{n-1}\cdots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_s)_{n-1}} a_n \frac{z^{n-1}}{(n-1)!},$$

Where $m \in N_0 = N \cup \{0\}$ and

$$\frac{D_{\lambda\mu}^m f_i(z)}{z} \ne 0, \text{ for all } z \in U.$$

On the other hand, we obtain

$$I'_{\lambda\mu}(f_1,...,f_m)(z)=\prod_{i=1}^{n}\left(\frac{D^m_{\lambda\mu}f_i(z)}{z}\right)^{\delta_i}$$

for $z \in$ U. This equality implies that

$$\ln I'_{\lambda\mu}(f_1,...,f_n)(z)=\sum_{i=1}^{n}\delta_i \ln\frac{D^m_{\lambda\mu}f_i(z)}{z}$$

or equivalently

$$\ln I'_{\lambda\mu}(f_1,...,f_n)(z)=\sum_{i=1}^{n}\delta_i[\ln D^m_{\lambda\mu}f_i(z)-\ln z].$$

By differentiating above equation, we get

$$\frac{I''_{\lambda\mu}(f_1,...,f_n)(z)}{I'_{\lambda\mu}(f_1,...,f_n)(z)}=\sum_{i=1}^{n}\delta_i\left[\frac{(D^m_{\lambda\mu}f_i(z))'}{D^m_{\lambda\mu}f_i(z)}-\frac{1}{z}\right].$$

After some calculation, we obtain

$$\left|z\frac{I''_{\lambda\mu}(f_1,...,f_n)(z)}{I'_{\lambda\mu}(f_1,...,f_n)(z)}\right|\le\sum_{i=1}^{n}|\delta_i|\left|\frac{z(D^m_{\lambda\mu}f_i(z))'}{D^m_{\lambda\mu}f_i(z)}-1\right|.$$

By hypothesis, since $\left|\frac{z(D^m_{\lambda\mu}f_i(z))'}{D^m_{\lambda\mu}f_i(z)}-1\right|\le 1$, $1 \le i \le n$, $z \in$ U, $0 \le \lambda \le \mu \le 1$, $m \in N_0 = N \cup \{0\}$

and since $|\delta_1|+|\delta_2|+\cdots+|\delta_n|\le 1$, we have

$$\left|z\frac{I''_{\lambda\mu}(f_1,...,f_n)(z)}{I'_{\lambda\mu}(f_1,...,f_n)(z)}\right|\le\sum_{i=1}^{n}|\delta_i|\le 1.$$

Hence, we obtain

$$(1-|z|^2)\left|z\frac{I''_{\lambda\mu}(f_1,...,f_n)(z)}{I'_{\lambda\mu}(f_1,...,f_n)(z)}\right|\le 1-|z^2|\le 1.$$

By Lemma 2.1, $I_{\lambda\mu}(f_1,f_2,...,f_n)(z)\in S.$

Remark 2. For $m = 0$, $D^0_{\lambda\mu}f_i(z)=D^0 f_i(z)=f_i(z)\in S$, $1 \le i \le n$, we have Theorem 1 in [4].

Corollary 2.4 Let m, $n \in N_0$, $\delta_i > 0$ and $f_i \in$ A, $1 \le i \le n$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$. If

$$\left|\frac{z(D^m_{\lambda}f_i(z))'}{D^m_{\lambda}f_i(z)}-1\right|\le 1,\quad (z\in \mathbb{U})$$

and $\sum_{i=1}^{n}|\delta_i|\le 1$, then $I_\lambda(f_1,f_2,\ldots,f_n)(z) \in$ S.

Corollary 2.5. Let m, $n \in N_0$, $\delta_i > 0$, $f_i \in$ A, $1 \le i \le n$, $\mu = 0$, $\lambda=1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$. If

$$\left|\frac{z(D^m f_i(z))'}{D^m f_i(z)}-1\right|\le 1,\quad (z\in \mathbb{U})$$

and $\sum_{i=1}^{n}|\delta_i|\le 1$, then $I(f_1,f_2,\ldots,f_n)(z) \in S$.

Theorem 2.6. Let m, $n \in N_0$, $\delta_i \in C$ and $f_i \in A$, $1 \le i \le n$ if

(i) $\left|D_{\lambda\mu}^{m}(f_i(z))\right| \le 1$

(ii) $\left|\frac{z^2(D_{\lambda\mu}^{m}(f_i(z))'}{(D_{\lambda\mu}^{m}(f_i(z))^2} - 1\right| \le 1, \quad (z \in \mathbb{U})$ and

(iii) $\sum_{i=1}^{n}|\delta_i| \le \frac{1}{3}$

then $I_{\lambda\mu}(f_1, f_2, \ldots, f_n)(z)$ defined in Definition 1 is univalent in U.

Proof. By the Lemma 2.1, we get

$$(1-|z|^2)\left|z\frac{I''_{\lambda\mu}(f_1,\ldots,f_n)(z)}{I'_{\lambda\mu}(f_1,\ldots,f_n)(z)}\right| \le (1-|z^2|)\sum_{i=1}^{n}|\delta_i|\left|\frac{z(D_{\lambda\mu}^{m}f_i(z))'}{D_{\lambda\mu}^{m}f_i(z)} - 1\right|$$

This inequality implies that

$$(1-|z|^2)\left|z\frac{I''_{\lambda\mu}(f_1,\ldots,f_n)(z)}{I'_{\lambda\mu}(f_1,\ldots,f_n)(z)}\right| \le (1-|z^2|)\sum_{i=1}^{n}\left[|\delta_i|\left|\frac{z(D_{\lambda\mu}^{m}f_i(z))'}{D_{\lambda\mu}^{m}f_i(z)}\right| + |\delta_i|\right]$$

$$= (1-|z^2|)\sum_{i=1}^{n}\left[|\delta_i|\left|\frac{z^2(D_{\lambda\mu}^{m}f_i(z))'}{(D_{\lambda\mu}^{m}f_i(z))^2}\right|\left|\frac{D_{\lambda\mu}^{m}f_i(z)}{|z|}\right| + |\delta_i|\right]$$

By Schwarz Lemma, we have

$$(1-|z|^2)\left|z\frac{I''_{\lambda\mu}(f_1,\ldots,f_n)(z)}{I'_{\lambda\mu}(f_1,\ldots,f_n)(z)}\right| \le (1-|z^2|)\sum_{i=1}^{n}\left[|\delta_i|\left|\frac{z^2(D_{\lambda\mu}^{m}f_i(z))'}{(D_{\lambda\mu}^{m}f_i(z))^2}\right| + |\delta_i|\right]$$

or

$$\begin{aligned}(1-|z|^2)\left|z\frac{I''_{\lambda\mu}(f_1,\ldots,f_n)(z)}{I'_{\lambda\mu}(f_1,\ldots,f_n)(z)}\right| &\le (1-|z^2|)\sum_{i=1}^{n}\left[|\delta_i|\left|\frac{z^2(D_{\lambda\mu}^{m}f_i(z))'}{(D_{\lambda\mu}^{m}f_i(z))^2} - 1 + 1\right| + |\delta_i|\right] \\ &\le (1-|z^2|)\sum_{i=1}^{n}|\delta_i|\left|\frac{z^2(D_{\lambda\mu}^{m}f_i(z))'}{(D_{\lambda\mu}^{m}f_i(z))^2} - 1\right| + 2|\delta_i| \\ &\le (1-|z^2|)\sum_{i=1}^{n}\left[|\delta_i| + 2|\delta_i|\right] \\ &\le 3(1-|z^2|)\sum_{i=1}^{n}|\delta_i| \\ &\le (1-|z^2|) \le 1, \quad \text{for all } z \in \mathbb{U}.\end{aligned}$$

So, by Lemma 2.1, $I_{\lambda\mu}(f_1, f_2, \ldots, f_n)(z) \in S$.

Remark 3. For $m = 0$, $n = 1$, $\delta_1 = \delta \in C$, $|\delta| \le 1/3$, $\delta_2 = \delta_3 = \cdots = \delta_n = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, we have Theorem 1 in [6].

Corollary 2.7. Let m, $n \in N_0$, $\delta_i > 0$ and $f_i \in A$, $1 \le i \le n$. If

(i) $\left|D^m_{\lambda\mu}(f_i(z))\right| \le 1$

(ii) $\left|\dfrac{z^2(D^m_{\lambda\mu}(f_i(z))'}{(D^m_{\lambda\mu}(f_i(z))^2}-1\right| \le 1, \quad (z \in \mathbb{U})$ and

(iii) $\delta_1 + \delta_2 + \cdots + \delta_n \le 1/3$

then $I_{\lambda\mu}(f_1, f_2, \ldots, f_n)(z) \in S$.

In [9] similar results were given by using the Ruscheweyh differential operator .

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