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On the Diophantine equation $4^x + p^y = z^2$

where p is a prime number

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Abstract

In this paper we show that all non-negative integer solutions of $\boldsymbol{4^x + p^y = z^2}$, ***where p is a prime number, are of the following*** $\boldsymbol{(x,p,y,z) \in \{(2,3,2,5)\} \cup \{(r, 2^{r+1}+1, 1, 2^r+1) : r \in \mathbb{N} \cup \{0\}\} \cup \{(r, 2, 2r+3, 3.2^r) : r \in \mathbb{N} \cup \{0\}\}}$.

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1. Introduction

D. Acu (2007) studied the Diophantine equation $2^x + 5^y = z^2$. He found that this equation has exactly two solutions in non-negative integer $(x, y, z) \in \{(3,0,3), (2,1,3)\}$. In 2010 the authors [5] studied the Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. They found that these equations have no non-negative integer solution. In 2011 A. Suvarnamani [6] studied solution of the Diophantine equation $2^x + p^y = z^2$ where p is a prime number. Unfortunately there is a misleading argument in [6] (see page 1417 line 15). This, for example, excludes the solutions $(x, y, z) = (3,1,5)$ and $(x, y, z) = (9,1,23)$ of the equation $2^x + 17^y = z^2$. Inspired by [5] and [6], we study the Diophantine equation $4^x + p^y = z^2$ where p is a prime number and x, y and z are non-negative integers.

2. Main Results

In this study, we use Catalan's conjecture (see [4]) stating that the only solution in integers $a > 1, b > 1, x > 1, y > 1$ of the equation $a^x - b^y = 1$ is $a = y = 3$ and $b = x = 2$.

First we consider the Diophantine equation $4^x + p^y = z^2$ where p is an odd prime number.

Theorem 1. *The Diophantine equation* $4^x + p^y = z^2$, *where p is an odd prime, has only non-negative integer solutions in the form* $(x, p, y, z) = (2,3,2,5)$ *or* $(x, p, y, z) = (r, 2^{r+1} + 1, 1, 2^r + 1)$ *where r is a non-negative integer.*

Proof. Case 1: $y = 0$. We have $4^x + 1 = z^2$. It easy to check that if $4^x + 1 = z^2$ has a solution, then $x \geq 2$ and z is an odd integer greater than 2. Let $z = 2k + 1$, for some integer $k \geq 1$. Thus

$$4^x + 1 = (2k+1)^2 = 4k^2 + 4k + 1.$$

It follows that $4^{x-1} = k^2 + k = k(k+1)$. We know that $k(k+1)$ has an odd factor greater than 1 and 4^{x-1} is not an odd factor. This is a contradiction. In this case the equation $4^x + p^y = z^2$ does not have a solution.

Case 2: $y > 0$. We have

$$p^y = z^2 - 4^x = (z + 2^x)(z - 2^x).$$

Then there are non-negative integers α, β such that $p^\alpha = z + 2^x, p^\beta = z - 2^x$, $\alpha > \beta$ and $\alpha + \beta = y$. Therefore,

$$p^{\beta}(p^{\alpha-\beta}-1) = p^{\alpha} - p^{\beta} = (z+2^x)-(z-2^x) = 2^{x+1}.$$

Since p is an odd prime, $\beta = 0$. This implies that $z = 2^x + 1, y = \alpha > 0$ and $p^{\alpha} - 2^{x+1} = 1$.

If $\alpha = 1$, then $p = 2^{x+1} + 1$. Thus $(x, p, y, z) = (x, 2^{x+1} + 1, 1, 2^x + 1)$ is a solution of $4^x + p^y = z^2$ whenever $2^{x+1} + 1$ is prime*.

If $\alpha > 1$, then $x > 0$. By Catalan's conjecture, we have $p = 3, \alpha = 2$ and $x = 2$. It follows that $z = 5$. In this case, $(x, p, y, z) = (2,3,2,5)$ is the only one solution.

It easy to check that $(x, p, y, z) = (2,3,2,5)$ or $(x, p, y, z) = (r, 2^{r+1} + 1, 1, 2^r + 1)$, where r is a non-negative integer, are solutions of $4^x + p^y = z^2$. This finishes the proof.

□

Finally, we consider the case $p = 2$, that is, the Diophantine equation $4^x + 2^y = z^2$. The following theorem is in [6], and it is rearranged for the reader's convenience.

Theorem 2. *Every non-negative integer solution of the Diophantine equation* $2^x + 2^y = z^2$ *where* $x \leq y$ *is of the form* $(x, y, z) = (2r-1, 2r-1, 2^r)$ *or* $(x, y, z) = (2r, 2r+3, 3.2^r)$ *where* r *is a non-negative integer.*

Proof. Let $z = a.2^r$ for some odd positive integer a and some non-negative integer r. Thus we have

$$2^x(1 + 2^{y-x}) = z^2 = a^2.2^{2r}.$$

If $x = y$, then $a = 1$ and $2^{x+1} = 2^{2r}$. This implies that $2r = x + 1$ or $x = 2r - 1$. Thus if $x = y$, every non-negative integer solution is of the form $(x, y, z) = (2r-1, 2r-1, 2^r)$ where r is a non-negative integer.

Next we consider the case $x < y$. Thus $2r = x$ and $a^2 - 2^{y-x} = 1$.
If $y - x = 1$, then $a^2 = 3$ which contradicts to the fact that a is an integer. Thus $y - x > 1$. By Catalan's conjecture, we have $a = 3, y - x = 3$. It follows that $y = 2r + 3$ and $z = 3.2^r$. Thus if $x < y$, every non-negative integer solution of the Diophantine $2^x + 2^y = z^2$ is of the form $(x, y, z) = (2r, 2r+3, 3.2^r)$ where r is a non-negative integer.

*We known that any prime number of the form $2^x + 1$ is called a Fermat prime. Until now 3, 5, 17, 257 and 65537 are only known Fermat primes. It remains unknown if there exist infinity many such prime numbers.

It is clear that $(x, y, z) = (2r-1, 2r-1, 2^r)$ or $(x, y, z) = (2r, 2r+3, 3.2^r)$, where r is non-negative integer, are solutions of the equation. The theorem is proved.

The Diophantine equation $4^x + 2^y = z^2$ is a special case of the Diophantine equation $2^x + 2^y = z^2$. It easy to verify the following corollary.

Corollary 3. *The non-negative solutions of the Diophantine equation* $4^x + 2^y = z^2$ *are of the form* $(x, y, z) = (r, 2r+3, 3.2^r)$, *where* r *is a non-negative integer.*

In conclude, the Diophantine equation $4^x + p^y = z^2$, where p is a prime number, is completely solved.

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