American. Jr. of Mathematics and Sciences
Vol. 1, No.1 ,(January 2012)

www.journalshub.com

ON THE DIOPHANTINE EQUATIONS OF $(2^n)^x + p^y = z^2$ TYPE

BILGE PEKER AND SELIN (INAG) CENBERCI

ABSTRACT. In this paper, we study on solutions of the Diophantine equations of $(2^n)^x + p^y = z^2$ type, when k,x, y, z, n are non-negative integers.

1. Introduction

There are lots of studies about the Diophantine equation of type $a^x + b^y = c^z$. In 1999, Z. Cao [3] proved that this equation has at most one solution with $z > 1$. In 2005, D. Acu [1] showed that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers, i.e. $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. J. Sandor [6] studied on the diophantine equation $4^x + 18^y = 22^z$. In 2011, A. Suvarnamani [9] consider the Diophantine equation $2^x + p^y = z^2$ where p is a prime and x, y, z are non-negative integers. A. Suvarnamani , A. Singta and S. Chotchaisthit [10] found solutions of the Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. In 1657 Frenicle de Bessy [5] solved a problem possed by Fermat: if p is an odd prime and $n \geq 2$ is an integer, then the equation $x^2 - 1 = p^n$ has no integer solution.

In this study, we gave solutions of the Diophantine equations $2^x + 19^y = z^2$, $8^x + 19^y = z^2$ and $8^x + 17^y = z^2$ when x, y, z are non-negative integers.

2. Preliminaries

In our study, we use Catalan's Conjecture ([4]) . Now we give this conjecture.

Conjecture 1. *(Catalan) The only solution in integers* $a > 1$, $b > 1$, $x > 1$ *and* $y > 1$ *of the equation* $a^x - b^y = 1$ *is* $a = y = 3$ *and* $b = x = 2$.

Now we give our theorems.

Theorem 1. *When k is a non-negative integer, solution of the Diophantine equation*

$$2^x + 19^y = z^2 \tag{1}$$

is given $(x, y, z) = (3, 0, 3)$.

Proof. If we write $x = 0$, then the Diophantine equation (1) becomes

$$1 + 19^y = z^2$$

and then, we have

$$z^2 - 1 = 19^y$$

i.e.

$$(z - 1)(z + 1) = 19^y$$

Date: October 02, 2012.
2000 *Mathematics Subject Classification.* Primary 11D61.
Key words and phrases. Exponential Diophantine Equation,

where $z-1=19^u$, $z+1=19^{y-u}$, $y>2u$. Then we obtain $19^{y-u}-19^u=2$ or $19^u(19^{y-2u}-1)=2$. If $u=0$, then $19^y-1=2$, i.e. $19^y=3$; which is impossible .

If we write $x=2k$, then the Diophantine equation (1) becomes

$$2^{2k}+19^y=z^2$$

and then, we get

$$z^2-2^{2k}=19^y$$

i.e.

$$(z-2^k)(z+2^k)=19^y$$

where $z-2^k=11^v$ and $z+2^k=19^{y-v}$, $y>2v$ and v is a non-negative integer. Then we get $19^{y-v}-19^v=2^{k+1}$ or $19^v(19^{y-2v}-1)=2^{k+1}$.

If $v=0$, we obtain $19^y-1=2^{k+1}$ or $19^y-2^{k+1}=1$. From $Catalan's$ $Conjecture$, it is obvious that we don't have a solution.

For $y=1$, we obtain $19-2^{k+1}=1$, i.e. $10=2^{k+1}$ which is impossible.

For $y=0$, we get $z^2-2^{2k}=1$, i.e. $z^2-2^{2k}=1$ which has no solution when $z=0$ or $z=1$. From $Catalan's$ $Conjecture$, $z=3$ and $2k=3$ which is impossible, too.

If we write $x=2k+1$, then the equation (1) becomes

$$2^{2k+1}+19^y=z^2$$

and then, we get

$$z^2-2^{2k+1}=19^y$$

i.e.

$$(z-2^{k+\frac{1}{2}})(z+2^{k+\frac{1}{2}})=19^y$$

where $z-2^{k+\frac{1}{2}}=19^u$ and $z+2^{k+\frac{1}{2}}=19^{y-u}$, $y>2u$ and u is a non-negative integer. Then we obtain $19^{y-u}-19^u=2^{k+\frac{3}{2}}$ or $19^u(19^{y-2u}-1)=2^{k+\frac{3}{2}}$. Clearly it is impossible if $u>0$.

If $u=0$, then we obtain $19^y-1=2^{k+\frac{3}{2}}$ or $19^y-2^{k+\frac{3}{2}}=1$. From $Catalan's$ $Conjecture$, it is obvious that we don't find a solution.

If $y=1$, then we get $19-2^{k+\frac{3}{2}}=1$, which is impossible, too.

If $y=0$, then we get $z^2-2^{2k+1}=1$ which has no solution when $z=0$ or $z=1$. By using $Catalan's$ $Conjecture$, there is only solution for $z=3$ and $2k+1=3$, i.e. $k=1$. So $(x,y,z)=(3,0,3)$. □

Theorem 2. *When k is a non-negative integer, the Diophantine equation*

$$8^x+19^y=z^2 \tag{2}$$

has no solution.

Proof. If $x = 2k$, then the equation (2) becomes

$$8^{2k} + 19^y = z^2$$

and then, we have

$$z^2 - 8^{2k} = 19^y$$

that is

$$(z - 8^k)(z + 8^k) = 19^y$$

where $z - 8^k = 19^v$ and $z + 8^k = 19^{y-v}$, $y > 2v$ and v is a non-negative integer. Then we get $19^v(19^{y-2v} - 1) = 2^{3k+1}$.

If $v = 0$, then we obtain $19^y - 1 = 2^{3k+1}$ or $19^y - 2^{3k+1} = 1$. From the $Catalan's$ $Conjecture$, it is obvious that, $y = 2$ and $3k + 1 = 3$ which is impossible.

For $y = 1$, we obtain $19 = 1 + 2^{3k+1}$. This does not give us a solution.

For $y = 0$, we get $z^2 - 8^{2k} = 1$, i.e. $z^2 - 2^{6k} = 1$ which is no solution when $z = 0$ or $z = 1$. From the $Catalan's$ $Conjecture$, $z = 3$ and $6k = 3$, which is impossible.

If we write $x = 2k + 1$, then the Diophantine equation (2) becomes

$$8^{2k+1} + 19^y = z^2$$

$$z^2 - 8^{2k+1} = 19^y$$

$$(z - 8^{k+\frac{1}{2}})(z + 8^{k+\frac{1}{2}}) = 19^y$$

where $z - 8^{k+\frac{1}{2}} = 19^u$ and $z + 8^{k+\frac{1}{2}} = 19^{y-u}$, $y > 2u$ and u is non-negative integer. Then we get $19^{y-u} - 19^u = 8^{k+\frac{3}{2}}$ or $19^u(19^{y-2u} - 1) = 8^{k+\frac{3}{2}}$.

If $u = 0$, then we obtain $19^y - 1 = 8^{k+\frac{3}{2}}$ or $19^y - 2^{3\left(k+\frac{3}{2}\right)} = 1$. From $Theorem$ 1, it is obvious that $y = 2$ and $3\left(k + \frac{3}{2}\right) = 3$ which is impossible.

If $y = 1$, then we get $19 - 8^{k+\frac{3}{2}} = 1$ i.e. $19 = 1 + 8^{k+\frac{3}{2}}$ which is impossible, too.

If $y = 0$, then we get $z^2 - 8^{k+\frac{3}{2}} = 1$ which has no solution when $z = 0$ or $z = 1$. $z^2 - 2^{3\left(k+\frac{3}{2}\right)} = 1$ is the Catalan's form. So $z = 3$ and $3\left(k + \frac{3}{2}\right) = 3$ which is impossible.

If $x = 0$, then the diophantine equation becomes

$$1 + 19^y = z^2$$

$$z^2 - 1 = 19^y$$

$$(z - 1)(z + 1) = 19^y$$

where $z - 1 = 19^v$, $z + 1 = 19^{y-v}$, $y > 2v$. Then we obtain $19^{y-v} - 19^v = 2$ or $19^v(19^{y-2v} - 1) = 2$. If $v = 0$, then $19^y - 1 = 2$, i.e. $19^y = 3$; which is impossible. This completes the proof of theorem. □

Theorem 3. *When k is a non-negative integer, solutions of the Diophantine equation*

$$8^x + 17^y = z^2 \tag{3}$$

is given with $(x, y, z) = (2, 1, 9)$.

Proof. If $x = 2k$, then the equation (3) becomes

$$8^{2k} + 17^y = z^2$$

and then, we have

$$z^2 - 8^{2k} = 17^y$$

that is

$$(z - 8^k)(z + 8^k) = 17^y$$

where $z - 8^k = 17^v$ and $z + 8^k = 17^{y-v}$, $y > 2v$ and v is a non-negative integer. Then we get $17^v(17^{y-2v} - 1) = 2^{3k+1}$.

If $v = 0$, then we obtain $17^y - 1 = 2^{3k+1}$ or $17^y - 2^{3k+1} = 1$. From the $Catalan's$ $Conjecture$, it is obvious that $y = 2$ and $3k + 1 = 3$ which is impossible.

For $y = 1$, we obtain $17 = 1 + 2^{3k+1}$. This gives us $k = 1$. So a solution of the diophantine equation (3) is $(x, y, z) = (2, 1, 9)$. For $y = 0$, we get $z^2 - 8^{2k} = 1$, i.e. $z^2 - 2^{6k} = 1$ which is no solution when $z = 0$ or $z = 1$. From the $Catalan's$ $Conjecture$, $z = 3$ and $6k = 3$ which is impossible too.

If we write $x = 2k + 1$, then the Diophantine equation (3) becomes

$$8^{2k+1} + 17^y = z^2$$

$$z^2 - 8^{2k+1} = 17^y$$

$$(z - 8^{k+\frac{1}{2}})(z + 8^{k+\frac{1}{2}}) = 17^y$$

where $z - 8^{k+\frac{1}{2}} = 17^u$ and $z + 8^{k+\frac{1}{2}} = 17^{y-u}$, $y > 2u$ and u is non-negative integer. Then we get $17^{y-u} - 17^u = 8^{k+\frac{3}{2}}$ or $17^u(17^{y-2u} - 1) = 8^{k+\frac{3}{2}}$.

If $u = 0$, then we obtain $17^y - 1 = 8^{k+\frac{3}{2}}$ or $17^y - 2^{3\left(k+\frac{3}{2}\right)} = 1$. From $Theorem$ 1, it is obvious that $y = 2$ and $3\left(k + \frac{3}{2}\right) = 3$ which is impossible.

If $y = 1$, then we get $17 - 8^{k+\frac{3}{2}} = 1$ i.e. $17 = 1 + 8^{k+\frac{3}{2}}$ which is impossible, too.

If $y = 0$, then we get $z^2 - 8^{k+\frac{3}{2}} = 1$ which has no solution when $z = 0$ or $z = 1$. $z^2 - 2^{3\left(k+\frac{3}{2}\right)} = 1$ is the Catalan's form. So $z = 3$ and $3\left(k + \frac{3}{2}\right) = 3$ which is impossible.

If $x = 0$, then the diophantine equation becomes

$$1 + 17^y = z^2$$

$$z^2 - 1 = 17^y$$

$$(z - 1)(z + 1) = 17^y$$

where $z - 1 = 17^v$, $z + 1 = 17^{y-v}$, $y > 2v$. Then we obtain $17^{y-v} - 17^v = 2$ or $17^v(17^{y-2v} - 1) = 2$. If $v = 0$, then $17^y - 1 = 2$, i.e. $17^y = 3$; which is impossible. This completes the proof of theorem. □

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Elementary Mathematics Education Programme, Ahmet Kelesoglu Education Faculty, Konya University.
E-mail address: `bilge.peker@yahoo.com`

Secondary MathematicsEducation Programme, Ahmet Kelesoglu Education Faculty, Konya University.
E-mail address: `inag_s@hotmail.com`