American. Jr. of Mathematics and Sciences
Vol. 1, No.1 ,(January 2012)

www.journalshub.com

A NOTE ON THE TENSOR FORM OF CHARACTERISTIC EQUATION OF 4×4 MATRIX USED IN HARISH-CHANDRA'S PAPER 'ALGEBRA OF DIRAC-MATRICES'

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Abstract

As the benifits of his beautiful formula for the product of Dirac-matrices Prof. Harish-Chandra evaluated the characteristic equation of a 4×4 matrix in his paper [1]. A trial to understand his ideas and his methods of calculation gives an internal pleasure at least to me. Through this paper I want to convey that pleasure to others also. The analysis given in this paper is useful in studying the relativistic wave equation of an electron.

Keywords: Characteristic polynomial, Characteristic equation, Trace, Dirac-matrices, Antisymmetric tensor.

AMS subject classification: 2010 MSC 11C20, 15A24.

1. INTRODUCTION:

Let T be any 4×4 matrix. The matrix, $(T-tI)$ where I is the 4×4 identity matrix and t is an indeterminate, is called the characteristic matrix of T. It's determinant $\Delta(t)=\det(T-tI)$, which is a polynomial in t, is called the characteristic polynomial of T and $\Delta(t)=\det(T-tI)=0$ is called the characteristic equation of T. The general characteristic polynomial of any 4×4 matrix T is of the form

$$\Delta(t)=t^4-a_1t^3+a_2t^2-a_3t+a_4, \tag{1}$$

where a_1,a_2,a_3,a_4 are constants s.t. $a_1=$ trace of $T=tr(T)$, $a_4=$ determinant value of $T=\det(T)$ and a_2,a_3 are the sum of the principal minors of order $2,3$ respectively.

If we put $t=0$ in equation (1) we get the constant a_4 i.e. $\det(T)$ and also if we differentiate equation (1), three times with respect to t, then put $t=0$ and divide by -6, we get the constant a_1 i.e. $tr(T)$.

Harish-Chandra Formula [1] for the product of Dirac-matrices can be given as

$$E_{\lambda\mu}E_{\alpha\beta}=-\delta_{\lambda\alpha}\delta_{\mu\beta}+\delta_{\mu\alpha}\delta_{\lambda\beta}+E_{\lambda\alpha}\delta_{\mu\beta}-E_{\mu\alpha}\delta_{\lambda\beta}-E_{\lambda\beta}\delta_{\mu\alpha}+E_{\mu\beta}\delta_{\lambda\alpha}-\frac{i}{2}\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}E^{\gamma\rho}, \tag{2}$$

where $E_{\lambda\mu}$ are Dirac matrices, $\delta_{\lambda\alpha}$ is a Kronecker delta function define as

$$\delta_{\lambda\alpha} = \begin{cases} 1, & \lambda = \alpha \\ 0, & \lambda \neq \alpha \end{cases}, \tag{3}$$

and $\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}$ is an antisymmetric tensor in all six indices with $\varepsilon_{012345} = 1$. All the indices

$\lambda, \mu, \nu, \alpha, \beta, \gamma, \rho$ vary from 0 to 5. If two or more than two indices value is same the quantity $\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}$ vanishes. Following paper [1] (2) is an equation in mixed tensor of rank four in a six dimensional space whose metric tensor is $\delta_{\lambda\alpha}$ for lowering or raising the index. The summation convention appear in a term if the same index appearing once above and below in the same term. For example

(i) $E_{\lambda}^{\mu} E_{\mu\alpha} = \sum_{\mu=0}^{5} E_{\lambda\mu} E_{\mu\alpha}$,

(ii) $E^{\lambda\mu} \varepsilon_{\lambda\mu\alpha\beta\gamma\rho} = \sum_{\lambda,\mu=0}^{5} E_{\lambda\mu} \varepsilon_{\lambda\mu\alpha\beta\gamma\rho}$,

(iii) but no summation is intended in term $E_{\lambda\mu} E_{\mu\alpha}$.

Dirac-matrices [1,2] $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ are the set of four matrices of order 4×4 whose entries are complex numbers, given by the relation

$$\gamma_\lambda \gamma_\mu + \gamma_\mu \gamma_\lambda = 2\delta_{\lambda\mu}, \qquad (\lambda, \mu = 1, 2, 3, 4) \tag{4}$$

these matrices are anticommute to each others with each square is identity matrix and generate a set of sixteen independent quantities which are also anticommute to each others (excluding identity matrix) can be given as

$1;\ \gamma_1, \gamma_2, \gamma_3, \gamma_4;\ \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_1\gamma_4, \gamma_2\gamma_3, \gamma_2\gamma_4, \gamma_3\gamma_4;\ \gamma_1\gamma_2\gamma_3, \gamma_1\gamma_2\gamma_4, \gamma_1\gamma_3\gamma_4, \gamma_2\gamma_3\gamma_4;\ \gamma_1\gamma_2\gamma_3\gamma_4$.

Since square of Dirac-matrices is 1 (identity matrix) but here some quantities square is -1. So if we multiply i by these quantities then all 15 quantities satisfy the equation (4) and we call all of them Dirac-matrices and which can be given as

$1; \gamma_1, \gamma_2, \gamma_3, \gamma_4;\ i\gamma_1\gamma_2, i\gamma_1\gamma_3, i\gamma_1\gamma_4, i\gamma_2\gamma_3, i\gamma_2\gamma_4, i\gamma_3\gamma_4;\ i\gamma_1\gamma_2\gamma_3, i\gamma_1\gamma_2\gamma_4, i\gamma_1\gamma_3\gamma_4, i\gamma_2\gamma_3\gamma_4;\ \gamma_1\gamma_2\gamma_3\gamma_4$.

For simplicity and quantities which can be used in equation (2), we can change above matrices in one index notation, suppose

$E_1 = i\gamma_1,\ E_2 = i\gamma_2,\ E_3 = i\gamma_3,\ E_4 = i\gamma_4$,

$E_5 = i(i\gamma_1\gamma_2), E_6 = i(i\gamma_1\gamma_3), E_7 = i(i\gamma_1\gamma_4), E_8 = i(i\gamma_2\gamma_3), E_9 = i(i\gamma_2\gamma_4), E_{10} = i(i\gamma_3\gamma_4)$,

$E_{11} = i(i\gamma_1\gamma_2\gamma_3), E_{12} = i(i\gamma_1\gamma_2\gamma_4), E_{13} = i(i\gamma_1\gamma_3\gamma_4), E_{14} = i(i\gamma_2\gamma_3\gamma_4)$,

$E_{15} = i(\gamma_1\gamma_2\gamma_3\gamma_4), E_{16} = i(1)$.

So equation (4) can be written as

$$E_\lambda E_\mu + E_\mu E_\lambda = -2\delta_{\lambda\mu}. \qquad (\lambda, \mu = 1 \;\; to \;\; 15,) \tag{5}$$

Hence $i,\ E_\lambda$ ($\lambda = 1,2,3,........,15$) form a complete set with properties $E_\lambda^2 = -1$ ($\lambda = 1 \;\; to \;\; 16$) and $E_\lambda E_\mu = -E_\mu E_\lambda$, ($\lambda, \mu = 1 \;\; to \;\; 15, \quad \lambda \neq \mu$). As in paper [1] we can change one index notation of these sixteen quantities into two indices notation with help of following rules

$$E_{\lambda\mu} = E_\lambda E_\mu, \qquad (\lambda \neq \mu) \qquad (\lambda = 0,1,2,3,4,5)$$

$$E_{\lambda\mu} = -E_{\mu\lambda}, \qquad (\lambda \neq \mu) \qquad (\lambda = 0,1,2,3,4,5)$$

$$E_\lambda = E_{0\lambda} = -E_{\lambda 0}, \qquad (\lambda = 1,2,3,4,5)$$

$$E_{\lambda\lambda} = 0. \qquad (\lambda = 0,1,2,3,4,5)$$

Then $i,\ E_{\lambda\mu}$ $(\lambda \neq \mu,\ \lambda, \mu = 0,1,2,3,4,5)$ form a complete set of Dirac-matrices with regard that two quantities which differ by a numerical factor $-1,\ i$ or $-i$ is essentially the same i.e.

$$i,\ E_{\lambda\mu} = \begin{Bmatrix} E_{00} & E_{01} & E_{02} & E_{03} & E_{04} & E_{05} \\ E_{10} & E_{11} & E_{12} & E_{13} & E_{14} & E_{15} \\ E_{20} & E_{21} & E_{22} & E_{23} & E_{24} & E_{25} \\ E_{30} & E_{31} & E_{32} & E_{33} & E_{34} & E_{35} \\ E_{40} & E_{41} & E_{42} & E_{43} & E_{44} & E_{45} \\ E_{50} & E_{51} & E_{52} & E_{53} & E_{54} & E_{55} \end{Bmatrix},$$

here the diagonal entries of $E_{\lambda\mu}$ are zero and entries below the diagonal are negative times corresponding entries in above the diagonal. So $i,\ E_{\lambda\mu}$ are consisting only sixteen distinct quantities $i; E_{01}, E_{02}, E_{03}, E_{04}, E_{05}; E_{12}, E_{13}, E_{14}, E_{15}; E_{23}, E_{24}, E_{25}; E_{34}, E_{35}; E_{45}$, which are anticommute to each others (excluding identity matrix i) and each square is -1, form a complete set of matrices of order 4×4. Hence these 16 matrices form a basis of a vector space of dimension 16 over the field of complex numbers. So any 4×4 matrix can be written as linear combination of these 16 matrices. This linear combination has been written in condensed form in next section with help of tensor calculus [3].

Prof. Harish-Chandra evaluated characteristic equation of any 4×4 matrix by using formula (2) with the help of two tensor identities [1]. The proof of these tensor identities have not given in his paper. I have verified these tensor identities and trying to prove them. These tensor identities can be given as

$$\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'}t_{\alpha'\beta'}t_{\gamma'\delta'} = 16\left(t_{\alpha\beta}t_{\gamma\delta} - t_{\alpha\gamma}t_{\beta\delta} - t_{\alpha\delta}t_{\gamma\beta}\right), \tag{6}$$

$$t_{\alpha\lambda}t_{\beta\rho}\varepsilon^{\alpha\beta\mu\nu\sigma\tau}t_{\mu\nu}t_{\sigma\tau} = \frac{1}{6}t_{\lambda\rho}\varepsilon^{\alpha\beta\mu\nu\sigma\tau}t_{\alpha\beta}t_{\mu\nu}t_{\sigma\tau}. \tag{7}$$

2. DISCUSSION:

Let T be any 4×4 matrix can be written [1] as

$$T = t + t_{\lambda\mu}E^{\lambda\mu}, \tag{8}$$

where $t, t_{\lambda\mu}$ are ordinary numbers commute to all other quantities s.t. $t_{\lambda\mu} = -t_{\mu\lambda}$ and $E_{\lambda\mu}$ are 15 Dirac-matrices and t represent identity matrix which multiply by number t. Now here we have first evaluated the characteristic polynomial of T. Equation (8) can be written as

$$T - t = t_{\lambda\mu}E^{\lambda\mu}, \tag{9}$$

squaring both side of above equation we get

$$(T-t)^2 = (t_{\lambda\mu}E^{\lambda\mu})^2,$$

$$= t_{\lambda\mu}E^{\lambda\mu}t_{\alpha\beta}E^{\alpha\beta} = \sum_{\lambda,\mu,\alpha,\beta=0}^{5} t_{\lambda\mu}t_{\alpha\beta}E_{\lambda\mu}E_{\alpha\beta} \qquad \text{(by using summation convention)}$$

$$= \sum_{\lambda,\mu,\alpha,\beta=0}^{5} t_{\lambda\mu}t_{\alpha\beta}\left\{-\delta_{\lambda\alpha}\delta_{\mu\beta} + \delta_{\mu\alpha}\delta_{\lambda\beta} + E_{\lambda\alpha}\delta_{\mu\beta} - E_{\mu\alpha}\delta_{\lambda\beta} - E_{\lambda\beta}\delta_{\mu\alpha} + E_{\mu\beta}\delta_{\lambda\alpha} - \frac{i}{2}\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}E^{\gamma\rho}\right\},$$

here we use well known Harish-Chandra Formula (2) for the product of Dirac-matrices.

Now $$\left(T-t\right)^2 = -\sum_{\lambda,\mu=0}^{5} t_{\lambda\mu}t_{\lambda\mu} + \sum_{\lambda,\mu=0}^{5} t_{\lambda\mu}t_{\mu\lambda} + \sum_{\lambda,\mu,\alpha=0}^{5} t_{\lambda\mu}t_{\alpha\mu}E_{\lambda\alpha} - \sum_{\lambda,\mu,\alpha=0}^{5} t_{\lambda\mu}t_{\alpha\lambda}E_{\mu\alpha}$$

$$- \sum_{\lambda,\mu,\beta=0}^{5} t_{\lambda\mu}t_{\mu\beta}E_{\lambda\beta} + \sum_{\lambda,\mu,\beta=0}^{5} t_{\lambda\mu}t_{\lambda\beta}E_{\mu\beta} - \frac{i}{2}\sum_{\lambda,\mu,\alpha,\beta=0}^{5} t_{\lambda\mu}t_{\alpha\beta}\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}E^{\gamma\rho},$$

by means of $t_{\mu\lambda} = -t_{\lambda\mu}$ in second term and opening summation in third, fourth, fifth, sixth term with using $E_{\lambda\mu} = -E_{\mu\lambda}$, these four term vanishes then so that

$$\left(T-t\right)^2 = -2\sum_{\lambda,\mu=0}^{5} t_{\lambda\mu}t_{\lambda\mu} - \frac{i}{2}\sum_{\lambda,\mu,\alpha,\beta=0}^{5} t_{\lambda\mu}t_{\alpha\beta}\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}E^{\gamma\rho} = -2t_{\lambda\mu}t^{\lambda\mu} - \frac{i}{2}t^{\lambda\mu}t^{\alpha\beta}\varepsilon_{\lambda\mu\alpha\beta\gamma\rho}E^{\gamma\rho},$$

for convenience change summation indices in the above equation then we above

$$\left(T-t\right)^2 + 2t_{\mu\nu}t^{\mu\nu} = -\frac{i}{2}t^{\alpha\beta}t^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}E^{\lambda\rho},$$

again squaring both side of above equation we get

$$\left\{\left(T-t\right)^2 + 2t_{\mu\nu}t^{\mu\nu}\right\}^2 = \left\{-\frac{i}{2}t^{\alpha\beta}t^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}E^{\lambda\rho}\right\}^2$$

$$=\left\{-\frac{i}{2}t^{\alpha\beta}t^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}E^{\lambda\rho}\right\}\left\{-\frac{i}{2}t^{\alpha'\beta'}t^{\gamma'\delta'}\varepsilon_{\alpha'\beta'\gamma'\delta'\mu\nu}E^{\mu\nu}\right\}$$

$$=\sum_{\lambda,\rho,\mu,\nu=0}^{5}\left\{-\frac{1}{4}t^{\alpha\beta}t^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}t^{\alpha'\beta'}t^{\gamma'\delta'}\varepsilon_{\alpha'\beta'\gamma'\delta'\mu\nu}E_{\lambda\rho}E_{\mu\nu}\right\}\qquad\text{(by using formula (2) in } E_{\lambda\rho}E_{\mu\nu}\text{)}$$

$$=\sum_{\lambda,\rho,\mu,\nu=0}^{5}\left\{-\frac{1}{4}t^{\alpha\beta}t^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}t^{\alpha'\beta'}t^{\gamma'\delta'}\varepsilon_{\alpha'\beta'\gamma'\delta'\mu\nu}\left\{\begin{matrix}-\delta_{\lambda\mu}\delta_{\rho\nu}+\delta_{\rho\mu}\delta_{\lambda\nu}+E_{\lambda\mu}\delta_{\rho\nu}-E_{\rho\mu}\delta_{\lambda\nu}\\ -E_{\lambda\nu}\delta_{\rho\mu}+E_{\rho\nu}\delta_{\lambda\mu}-\frac{i}{2}\varepsilon_{\lambda\rho\mu\nu\sigma\tau}E^{\sigma\tau}\end{matrix}\right\}\right\}$$

$$=-\frac{1}{4}t^{\alpha\beta}t^{\gamma\delta}t^{\alpha'\beta'}t^{\gamma'\delta'}\left\{-\sum_{\lambda,\rho=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\lambda\rho}+\sum_{\lambda,\rho=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\rho\lambda}+\sum_{\lambda,\rho,\mu=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\mu\rho}E_{\lambda\mu}\right.$$

$$-\sum_{\lambda,\rho,\mu=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\mu\lambda}E_{\rho\mu}-\sum_{\lambda,\rho,\nu=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\rho\nu}E_{\lambda\nu}+\sum_{\lambda,\rho,\nu=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\lambda\nu}E_{\rho\nu}$$

$$\left.-\frac{i}{2}\sum_{\lambda,\rho,\mu,\nu=0}^{5}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon_{\alpha'\beta'\gamma'\delta'\mu\nu}\varepsilon_{\lambda\rho\mu\nu\sigma\tau}E^{\sigma\tau}\right\},$$

we used summation convention in 1st, 2nd, 7th terms and 3rd, 4th terms cancel out the terms 5th, 6th respectively in the above equation then

$$\left\{(T-t)^2+2t_{\mu\nu}t^{\mu\nu}\right\}^2$$

$$=-\frac{1}{4}t^{\alpha\beta}t^{\gamma\delta}(-2)\ \varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\ \varepsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'}\ t_{\alpha'\beta'}t_{\gamma'\delta'}+\frac{i}{8}\ \varepsilon^{\alpha\beta\gamma\delta\lambda\rho}\ t_{\alpha\beta}t_{\gamma\delta}\ \varepsilon^{\alpha'\beta'\gamma'\delta'\mu\nu}\ t_{\alpha'\beta'}t_{\gamma'\delta'}\ \varepsilon_{\lambda\rho\mu\nu\sigma\tau}E^{\sigma\tau}$$

$$=\frac{1}{2}t^{\alpha\beta}t^{\gamma\delta}\ \varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\ \varepsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'}\ t_{\alpha'\beta'}t_{\gamma'\delta'}+\frac{i}{8}\ \varepsilon^{\alpha\beta\mu\nu\sigma\tau}\ t_{\mu\nu}t_{\sigma\tau}\ \varepsilon^{\gamma\delta\mu'\nu'\sigma'\tau'}\ t_{\mu'\nu'}t_{\sigma'\tau'}\ \varepsilon_{\alpha\beta\gamma\delta\lambda\rho}E^{\lambda\rho},$$

for convenience here we changed summation indices in second term of right side of equation then above equation can be written as

$$\left\{(T-t)^2+2t_{\mu\nu}t^{\mu\nu}\right\}^2-\frac{1}{2}t^{\alpha\beta}t^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta\lambda\rho}\varepsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'}\ t_{\alpha'\beta'}t_{\gamma'\delta'}$$

$$=\frac{i}{8}\ \varepsilon^{\alpha\beta\mu\nu\sigma\tau}\ t_{\mu\nu}t_{\sigma\tau}\ \varepsilon^{\gamma\delta\mu'\nu'\sigma'\tau'}t_{\mu'\nu'}t_{\sigma'\tau'}\varepsilon_{\alpha\beta\lambda\rho\gamma\delta}E^{\lambda\rho}$$

$$=\frac{i}{8}\ \varepsilon^{\alpha\beta\mu\nu\sigma\tau}\ t_{\mu\nu}t_{\sigma\tau}\left\{\varepsilon_{\alpha\beta\lambda\rho\gamma\delta}\varepsilon^{\gamma\delta\mu'\nu'\sigma'\tau'}t_{\mu'\nu'}t_{\sigma'\tau'}\right\}E^{\lambda\rho}\qquad\text{(using first tensor identity (6))}$$

$$=\frac{i}{8}\ \varepsilon^{\alpha\beta\mu\nu\sigma\tau}\ t_{\mu\nu}t_{\sigma\tau}\left\{16\left\{t_{\alpha\beta}t_{\lambda\rho}-t_{\alpha\lambda}t_{\beta\rho}-t_{\alpha\rho}t_{\lambda\beta}\right\}\right\}E^{\lambda\rho}$$

$$= 2i\varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau} \left\{ t_{\alpha\beta} t_{\lambda\rho} E^{\lambda\rho} - 2t_{\alpha\lambda} t_{\beta\rho} E^{\lambda\rho} \right\}$$ (by using equation (9))

$$= 2i\varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} (T-t) - 4it_{\alpha\lambda} t_{\beta\rho} \varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau} E^{\lambda\rho}$$

So $\left\{(T-t)^2 + 2t_{\mu\nu} t^{\mu\nu}\right\}^2 - \frac{1}{2} t^{\alpha\beta} t^{\gamma\delta} \varepsilon_{\alpha\beta\gamma\delta\lambda\rho} \varepsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'} t_{\alpha'\beta'} t_{\gamma'\delta'}$

$$= 2i\varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} (T-t) - 4it_{\alpha\lambda} t_{\beta\rho} \varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau} E^{\lambda\rho} . \qquad (10)$$

But the quantity $-4it_{\alpha\lambda} t_{\beta\rho} \varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau} E^{\lambda\rho}$

$$= -4i \left\{ \frac{1}{6} t_{\lambda\rho} \varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} \right\} E^{\lambda\rho}$$ (using second tensor identity (7))

$$= -\frac{4}{6} i\varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} t_{\lambda\rho} E^{\lambda\rho}$$ (again using equation (9))

$$= -\frac{2}{3} i\varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} (T-t) .$$

Also the quantity $-\frac{1}{2} t^{\alpha\beta} t^{\gamma\delta} \varepsilon_{\alpha\beta\gamma\delta\lambda\rho} \varepsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'} t_{\alpha'\beta'} t_{\gamma'\delta'}$ (again using first tensor identity (6))

$$= -\frac{1}{2} t^{\alpha\beta} t^{\gamma\delta} \left\{ 16 \left(t_{\alpha\beta} t_{\gamma\delta} - t_{\alpha\gamma} t_{\beta\delta} - t_{\alpha\delta} t_{\gamma\beta} \right) \right\}$$

$$= -8t^{\alpha\beta} t^{\gamma\delta} \left(t_{\alpha\beta} t_{\gamma\delta} - t_{\alpha\gamma} t_{\beta\delta} - t_{\alpha\delta} t_{\gamma\beta} \right)$$

$$= -8 \left\{ \left(t^{\alpha\beta} t_{\alpha\beta} \right)^2 - t^{\alpha\beta} \left(-t^{\gamma\delta} \right) \left(-t_{\delta\alpha} \right) t_{\beta\gamma} - t^{\alpha\beta} t^{\gamma\delta} \left(-t_{\delta\alpha} \right) \left(-t_{\beta\gamma} \right) \right\}$$

$$= -8 \left\{ \left(t^{\alpha\beta} t_{\alpha\beta} \right)^2 - t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha} - t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha} \right\}$$

$$= -8 \left\{ \left(t^{\alpha\beta} t_{\alpha\beta} \right)^2 - 2t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha} \right\} .$$

Substituting these values in equation (10) so that

$$\left\{ \left(T-t \right)^2 + 2t_{\mu\nu} t^{\mu\nu} \right\}^2 - 8 \left\{ \left(t^{\alpha\beta} t_{\alpha\beta} \right)^2 - 2t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha} \right\} = \frac{4}{3} i\varepsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} \left(T-t \right) ,$$

for convenience again changed the summation indices in right side of equation as

$$\left\{ \left(T-t \right)^2 + 2t_{\mu\nu} t^{\mu\nu} \right\}^2 - 8 \left\{ \left(t^{\alpha\beta} t_{\alpha\beta} \right)^2 - 2t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha} \right\} = \frac{4}{3} it_{\alpha\beta} t_{\gamma\delta} t_{\lambda\rho} \varepsilon^{\alpha\beta\gamma\delta\lambda\rho} \left(T-t \right) . \qquad (11)$$

So Characteristic polynomial of matrix T can be

$$\Delta(T)=\left\{\left(T-t\right)^2+2t_{\mu\nu}t^{\mu\nu}\right\}^2-8\left\{\left(t^{\alpha\beta}t_{\alpha\beta}\right)^2-2t^{\alpha\beta}t_{\beta\gamma}t^{\gamma\delta}t_{\delta\alpha}\right\}-\frac{4}{3}it_{\alpha\beta}t_{\gamma\delta}t_{\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta\lambda\rho}\left(T-t\right), \quad (12)$$

also Characteristic equation of matrix T is

$$\left\{\left(T-t\right)^2+2t_{\mu\nu}t^{\mu\nu}\right\}^2-8\left\{\left(t^{\alpha\beta}t_{\alpha\beta}\right)^2-2t^{\alpha\beta}t_{\beta\gamma}t^{\gamma\delta}t_{\delta\alpha}\right\}-\frac{4}{3}it_{\alpha\beta}t_{\gamma\delta}t_{\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta\lambda\rho}\left(T-t\right)=0, \quad (13)$$

and put $T=0$ in right side of equation (12) we get the determinant value of matrix T as

$$\det(T)=\left\{t^2+2t_{\mu\nu}t^{\mu\nu}\right\}^2-8\left\{\left(t^{\alpha\beta}t_{\alpha\beta}\right)^2-2t^{\alpha\beta}t_{\beta\gamma}t^{\gamma\delta}t_{\delta\alpha}\right\}+\frac{4}{3}it_{\alpha\beta}t_{\gamma\delta}t_{\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta\lambda\rho}t. \quad (14)$$

Now for trace of matrix T,

since $\left\{(T-t)^2+2t_{\mu\nu}t^{\mu\nu}\right\}^2=\left\{T^2+t^2-2Tt+2t_{\mu\nu}t^{\mu\nu}\right\}^2$

$$=T^4+t^4+4T^2t^2+4\left(t_{\mu\nu}t^{\mu\nu}\right)^2+2T^2t^2-4T^3t+4T^2\left(t_{\mu\nu}t^{\mu\nu}\right)$$

$$-4Tt^3+4t^2\left(t_{\mu\nu}t^{\mu\nu}\right)-8Tt\left(t_{\mu\nu}t^{\mu\nu}\right),$$

so equation (12) becomes

$$\Delta\left(T\right)=T^4+t^4+4T^2t^2+4\left(t_{\mu\nu}t^{\mu\nu}\right)^2+2T^2t^2-4T^3t+4T^2\left(t_{\mu\nu}t^{\mu\nu}\right)-4Tt^3+4t^2\left(t_{\mu\nu}t^{\mu\nu}\right)$$

$$-8Tt\left(t_{\mu\nu}t^{\mu\nu}\right)-8\left\{\left(t^{\alpha\beta}t_{\alpha\beta}\right)^2-2t^{\alpha\beta}t_{\beta\gamma}t^{\gamma\delta}t_{\delta\alpha}\right\}-\frac{4}{3}it_{\alpha\beta}t_{\gamma\delta}t_{\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta\lambda\rho}\left(T-t\right),$$

differentiate above equation three times with respect to T then put $T=0$ and divide by -6 we get trace of matrix T.

$$\frac{d^3\Delta(T)}{dT^3}=24T-24t, \quad \left[\frac{d^3\Delta(T)}{dT^3}\right]_{T=0}=0-24t, \quad \therefore\ tr(T)=\frac{1}{-6}\left[\frac{d^3(\Delta(T))}{dT^3}\right]_{T=0}=4t \quad (15)$$

equation (15) agrees with the result given in papers [1].

3. CONCLUSION:

The above reckoning helps in studying the whole paper [1] and gives a new method to find out the characteristic equation and determinant value of any 4×4 matrix in tensor form with the help of Harish-Chandra Formula for the product of Dirac-matrices.

ACKNOWLEDGEMENT:

I am grateful to my supervisor Dr. T. N. Trivedi for guidance and encouragement during the study of this work.

I am also thankful to CSIR, New Delhi, India for providing financial support.

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