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HERMITE-HADAMARD TYPE INEQUALITIES FOR CO-ORDINATED HARMONICALLY CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this article, we established the Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions via generalized fractional integrals known as Katugampola fractional integrals. Through these result we also obtained the Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions via Riemann Liouville fractional integrals.

1. Introduction

For convex function $f : I \to \mathbb{R}$ on an interval of real line, for all $a, b \in I$ and $t \in [0, 1]$, the Hermite-Hadamard inequality [4], is given as

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Dragomir [3] gave the Hadamard's inequality for co-ordinate convex functions which is defined as:

Definition 1.1 ([3]). *A function $f : \Delta = [a, b]\times[c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called co-ordinate convex on Δ with $a < b$ and $c < d$ if the partial functions*

$$f_y : [a, b] \to \mathbb{R},\ f_y(u) = f(u, y) \text{ and } f_x : [c, d] \to \mathbb{R},\ f_x(v) = f(x, v),$$

are convex for all $x \in [a, b]$ and $y \in [c, d]$.

While Sarikaya [10], gave Hermite-Hadamard type inequalities for co-ordinated convex functions via fractional integrals (for more results and details see [1]-[3] and [8]-[10]), defined it as:

Definition 1.2 ([10]). *A function $f : \Delta = [a, b]\times[c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called coordinate convex on Δ with $a < b$ and $c < d$ if the following inequality holds:*

$$\begin{aligned} &f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\leq t\lambda f(x, y) + t(1-\lambda)f(x, w) + (1-t)\lambda f(z, y) + (1-t)(1-\lambda)f(z, w), \end{aligned}$$

Key words and phrases. Hermite-Hadamard inequalities; Riemann Liouville fractional integral; Katugampola fractional integral; harmonically convex functions on co-ordinates.

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for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in \Delta$.

Definition 1.3 ([8]). *A function $f : \Delta = [a,b] \times [c,d] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is called coordinated harmonically convex on Δ with $a < b$ and $c < d$ if*

$$\begin{aligned}
&f\left(\frac{xz}{tx+(1-t)z}, \frac{yw}{\lambda y+(1-\lambda)w}\right) \\
&\leq t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w),
\end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in \Delta$.

Clearly, a function $f : \Delta = [a,b]\times[c,d] \subseteq (0,\infty)\times(0,\infty) \to \mathbb{R}$ is called coordinate harmonically convex on Δ with $a < b$ and $c < d$ if the partial functions

$$f_y : [a,b] \to \mathbb{R},\ f_y(u) = f(u,y) \text{ and } f_x : [c,d] \to \mathbb{R},\ f_x(v) = f(x,v)$$

are harmonically convex for all $x \in [a,b]$ and $y \in [c,d]$ (see [8] and [9] for more results).

Theorem 1.4 ([8]). *Let $f : \Delta = [a,b]\times[c,d] \subseteq (0,\infty)\times(0,\infty) \to \mathbb{R}$ be co-ordinated harmonically convex on Δ with $a < b$ and $c < d$. Then*

$$\begin{aligned}
f\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) &\leq \frac{(ab)(cd)}{(b-a)(d-c)} \int_a^b \int_c^d \frac{f(x,y)}{x^2y^2} dydx \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{2}$$

Definition 1.5 ([6]). *Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha(>0)$ of $f \in X_c^p(a,b)$ are defined by,*

$${}^{\rho}I_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} f(t)dt,$$

and

$${}^{\rho}I_{b-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^\rho - x^\rho)^{\alpha-1} t^{\rho-1} f(t)dt,$$

with $a < x < b$ and $\rho > 0$. Where $X_c^p(a,b)$ ($c \in \mathbb{R}, 1 \leq p \leq \infty$) is the space of those complex valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by,

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$ and for the case $p = \infty$,

$$\|f\|_{X_c^\infty} = ess\ sup_{a \leq t \leq b}[t^c|f(t)|].$$

Definition 1.6. *Let $f \in L_1([a,b] \times [c,d])$. The Katugampola fractional integrals ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{a+,c+}$, ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{a+,d-}$, ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{b-,c+}$ and ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{b-,d-}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by*

$$
{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{a+,c+}f(x,y) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x^{\rho_1} - t^{\rho_1})^{\alpha-1}(y^{\rho_2} - s^{\rho_2})^{\beta-1} t^{\rho_1-1} s^{\rho_2-1} f(t,s)dsdt,
$$

with $x > a$ $y > c$,

$$
{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{a+,d-}f(x,y) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x^{\rho_1} - t^{\rho_1})^{\alpha-1}(s^{\rho_2} - y^{\rho_2})^{\beta-1} t^{\rho_1-1} s^{\rho_2-1} f(t,s)dsdt,
$$

with $x > a$ $y < d$,

$$
{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{b-,c+}f(x,y) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t^{\rho_1} - x^{\rho_1})^{\alpha-1}(y^{\rho_2} - s^{\rho_2})^{\beta-1} t^{\rho_1-1} s^{\rho_2-1} f(t,s)dsdt,
$$

with $x < b$ $y > c$, and

$$
{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{b-,d-}f(x,y) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t^{\rho_1} - x^{\rho_1})^{\alpha-1}(s^{\rho_2} - y^{\rho_2})^{\beta-1} t^{\rho_1-1} s^{\rho_2-1} f(t,s)dsdt,
$$

with $x < b$ $y < d$, respectively. Where the Gamma function Γ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

We recall classical Beta function and classical Hypergeometric function:

$$
\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,
$$

and

$$
{}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \ c > b > 0, \ |z| < 1.
$$

Now for $\rho > 0$, we generalize these functions as

$$
{}^{\rho}\gamma(a,b) = \int_0^1 (x^\rho)^{a-1}(1-x^\rho)^{b-1}x^{\rho-1}dx,
$$

and

$$
{}^{\rho}_{2}G_1(a,b;c;z) = \frac{1}{{}^{\rho}\gamma(b,c-b)} \int_0^1 (t^\rho)^{b-1}(1-t^\rho)^{c-b-1}(1-zt^\rho)^{-a}t^{\rho-1}dt, \ c > b > 0, \ |z| < 1.
$$

Clearly as $\rho \to 1$, then generalized Beta function and generalized Hypergeometric function becomes the classical Beta function and the Hypergeometric function, respectively. In this paper, we prove some Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions via Katugampola fractional integrals.

2. Results

First we prove the following Hermite-Hadamard ineqality for co-ordinated harmonically convex functions in Katugampola fractional integrals.

Theorem 2.1. *Let $\alpha, \beta > 0$ and $\rho_1, \rho_2 > 0$. Let $f : \Delta = [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a function with $0 < a < b$, $0 < c < d$. If f is also co-oordinated harmonically convex on Δ with $a < b$ and $c < d$ and $f \in L_1(\Delta)$(i.e., f is Lebesgue integrable over Δ). Then*

(3)

$$\begin{aligned}
&f\left(\frac{2a^{\rho_1}b^{\rho_1}}{a^{\rho_1}+b^{\rho_1}}, \frac{2c^{\rho_2}d^{\rho_2}}{c^{\rho_2}+d^{\rho_2}}\right) \\
&\leq \frac{\rho_1^{\alpha}\rho_2^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\Big[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-} f\circ g(\frac{1}{b^{\rho_1}}, \frac{1}{d^{\rho_2}}) \\
&+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+} f\circ g(\frac{1}{b^{\rho_1}}, \frac{1}{c^{\rho_2}}) +\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-} f\circ g(\frac{1}{a^{\rho_1}}, \frac{1}{d^{\rho_2}}) +\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+} f\circ g(\frac{1}{a^{\rho_1}}, \frac{1}{c^{\rho_2}})\Big] \\
&\leq \frac{f(a^{\rho_1},c^{\rho_2}) + f(a^{\rho_1},d^{\rho_2}) + f(b^{\rho_1},c^{\rho_2}) + f(b^{\rho_1},d^{\rho_2})}{4},
\end{aligned}$$

where $g(x^{\rho_1}, y^{\rho_2}) = \left(\frac{1}{x^{\rho_1}}, \frac{1}{y^{\rho_2}}\right)$.

Proof. Let $(x^{\rho_1}, y^{\rho_2}), (z^{\rho_1}, w^{\rho_2}) \in \Delta$ and $t, \lambda \in [0,1]$. Since f is co-ordinated harmonically convex on Δ, we have

$$\begin{aligned}
&f\left(\frac{x^{\rho_1}z^{\rho_1}}{t^{\rho_1}x^{\rho_1}+(1-t^{\rho_1})z^{\rho_1}}, \frac{y^{\rho_2}w^{\rho_2}}{\lambda^{\rho_2}y^{\rho_2}+(1-\lambda^{\rho_2})w^{\rho_2}}\right) \\
&\leq t^{\rho_1}\lambda^{\rho_2}f(x^{\rho_1},y^{\rho_2}) + t^{\rho_1}(1-\lambda^{\rho_2})f(x^{\rho_1},w^{\rho_2}) \\
&+ (1-t^{\rho_1})\lambda^{\rho_1}f(z^{\rho_1},y^{\rho_2}) + (1-t^{\rho_1})(1-\lambda^{\rho_2})f(z^{\rho_1},w^{\rho_2}).
\end{aligned} \tag{4}$$

By taking $x^{\rho_1} = \frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}$, $z^{\rho_1} = \frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}}$, $y^{\rho_2} = \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}$, $w^{\rho_2} = \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}$ and $t^{\rho_1} = \lambda^{\rho_2} = \frac{1}{2}$ in (4), we get

$$\begin{aligned}
&f\left(\frac{2a^{\rho_1}b^{\rho_1}}{a^{\rho_1}+b^{\rho_1}}, \frac{2c^{\rho_2}d^{\rho_2}}{c^{\rho_2}+d^{\rho_2}}\right) \\
&\leq \frac{1}{4}\Big[f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right) \\
&+ f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right) \\
&+ f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right) \\
&+ f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right)\Big].
\end{aligned} \tag{5}$$

Multiplying both sides of (5) by $t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1}$ and then integrating with respect to (t,λ) over $[0,1]\times[0,1]$, we get

$$\begin{aligned}
&\frac{1}{\rho_1\rho_2\alpha\beta} f\left(\frac{2a^{\rho_1}b^{\rho_1}}{a^{\rho_1}+b^{\rho_1}}, \frac{2c^{\rho_2}d^{\rho_2}}{c^{\rho_2}+d^{\rho_2}}\right)\\
&\leq \frac{1}{4}\Big[\int_0^1\int_0^1 f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right) t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1}dtd\lambda\\
&+\int_0^1\int_0^1 f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right) t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1}dtd\lambda\\
&+\int_0^1\int_0^1 f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right) t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1}dtd\lambda\\
&+\int_0^1\int_0^1 f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right) t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1}dtd\lambda\Big].
\end{aligned} \tag{6}$$

Applying change of variable, we find

$$\begin{aligned}
&f\left(\frac{2a^{\rho_1}b^{\rho_1}}{a^{\rho_1}+b^{\rho_1}}, \frac{2c^{\rho_2}d^{\rho_2}}{c^{\rho_2}+d^{\rho_2}}\right) \leq \frac{\rho_1\rho_2\alpha\beta}{4}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\\
&\times\Big[\int_{1/d}^{1/c}\int_{1/b}^{1/a}\left(\frac{1}{a^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{c^{\rho_2}}-y^{\rho_2}\right)^{\beta-1} x^{\rho_1-1}y^{\rho_2-1} f\left(\frac{1}{x^{\rho_1}}, \frac{1}{y^{\rho_2}}\right)dxdy\\
&+\int_{1/d}^{1/c}\int_{1/b}^{1/a}\left(\frac{1}{a^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{d^{\rho_2}}\right)^{\beta-1} x^{\rho_1-1}y^{\rho_2-1} f\left(\frac{1}{x^{\rho_1}}, \frac{1}{y^{\rho_2}}\right)dxdy\\
&+\int_{1/d}^{1/c}\int_{1/b}^{1/a}\left(x^{\rho_1}-\frac{1}{b^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{c^{\rho_2}}-y^{\rho_2}\right)^{\beta-1} x^{\rho_1-1}y^{\rho_2-1} f\left(\frac{1}{x^{\rho_1}}, \frac{1}{y^{\rho_2}}\right)dxdy\\
&+\int_{1/d}^{1/c}\int_{1/b}^{1/a}\left(x^{\rho_1}-\frac{1}{b^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{d^{\rho_2}}\right)^{\beta-1} x^{\rho_1-1}y^{\rho_2-1} f\left(\frac{1}{x^{\rho_1}}, \frac{1}{y^{\rho_2}}\right)dxdy\Big].
\end{aligned} \tag{7}$$

Then by multiplying and dividing by $\rho_1^{1-\alpha}\rho_2^{1-\beta}\Gamma(\alpha)\Gamma(\beta)$ on right hand side of inequality (7), we get the first inequality of (3). For the second inequality of (3) we use the co-ordinated harmonically convexity of f as:

$$\begin{aligned}
&f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right)\\
&\leq t^{\rho_1}\lambda^{\rho_2}f(a^{\rho_1},c^{\rho_2})+t^{\rho_1}(1-\lambda^{\rho_2})f(a^{\rho_1},d^{\rho_2})\\
&+(1-t^{\rho_1})\lambda^{\rho_1}f(b^{\rho_1},c^{\rho_2})+(1-t^{\rho_1})(1-\lambda^{\rho_2})f(b^{\rho_1},d^{\rho_2}),
\end{aligned}$$

$$\begin{aligned}
&f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}}, \frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right)\\
&\leq t^{\rho_1}\lambda^{\rho_2}f(a^{\rho_1},d^{\rho_2})+t^{\rho_1}(1-\lambda^{\rho_2})f(a^{\rho_1},c^{\rho_2})\\
&+(1-t^{\rho_1})\lambda^{\rho_1}f(b^{\rho_1},d^{\rho_2})+(1-t^{\rho_1})(1-\lambda^{\rho_2})f(b^{\rho_1},c^{\rho_2}),
\end{aligned}$$

$$
\begin{aligned}
&f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}},\frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right)\\
&\le t^{\rho_1}\lambda^{\rho_2}f(b^{\rho_1},c^{\rho_2})+t^{\rho_1}(1-\lambda^{\rho_2})f(b^{\rho_1},d^{\rho_2})\\
&+(1-t^{\rho_1})\lambda^{\rho_1}f(a^{\rho_1},c^{\rho_2})+(1-t^{\rho_1})(1-\lambda^{\rho_2})f(a^{\rho_1},d^{\rho_2}),
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}},\frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right)\\
&\le t^{\rho_1}\lambda^{\rho_2}f(b^{\rho_1},d^{\rho_2})+t^{\rho_1}(1-\lambda^{\rho_2})f(b^{\rho_1},c^{\rho_2})\\
&+(1-t^{\rho_1})\lambda^{\rho_1}f(a^{\rho_1},d^{\rho_2})+(1-t^{\rho_1})(1-\lambda^{\rho_2})f(a^{\rho_1},c^{\rho_2}).
\end{aligned}
$$

Thus by adding above inequalities, we get

$$
\begin{aligned}
&f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}},\frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right)\\
&+f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}},\frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right)\\
&+f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}},\frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}}\right)\\
&+f\left(\frac{a^{\rho_1}b^{\rho_1}}{t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}},\frac{c^{\rho_2}d^{\rho_2}}{\lambda^{\rho_2}d^{\rho_2}+(1-\lambda^{\rho_2})c^{\rho_2}}\right)\\
&\le f(a^{\rho_1},c^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},d^{\rho_2}).
\end{aligned}
\tag{8}
$$

Thus by multiplying (8) by $t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1}$ and then integrating with respect to (t,λ) over $[0,1]\times[0,1]$, we get the second inequality of (3). Hence the proof is completed. □

Remark 2.2. *In inequality* (3)*, if one takes* $\rho_1=\rho_2=1$*,* $\alpha=\beta=1$ *and using change of variable* $u=1/x$ *and* $v=1/y$*, then one has inequality* (2)*.*

Lemma 2.3. *Let* $f:\Delta=[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a partial differentiable mapping on* Δ *with* $0<a<b$ *and* $0<c<b$*. If* $\partial^2f/\partial t\partial\lambda\in L_1(\Delta)$*,*

then following equality holds:

(9)

$$\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}$$
$$\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\left[\,^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+\,^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\right.$$
$$\left.+\,^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+\,^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\right]-\Xi$$
$$=\frac{\rho_1\rho_2a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_1}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4}\left[\int_0^1\int_0^1\frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)d\lambda dt\right.$$
$$-\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)d\lambda dt-\int_0^1\int_0^1\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}$$
$$\left.\times\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)d\lambda dt+\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)d\lambda dt\right],$$

where

(10)

$$\Xi=\frac{\rho_1^{\alpha+1}\Gamma(\alpha+1)}{4}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left[\,^{\rho_1}I^{\alpha}_{1/b+}f\circ g_1\left(\frac{1}{a^{\rho_1}},d^{\rho_2}\right)+\,^{\rho_1}I^{\alpha}_{1/b+}f\circ g_1\left(\frac{1}{a^{\rho_1}},c^{\rho_2}\right)\right.$$
$$\left.+\,^{\rho_1}I^{\alpha}_{1/a-}f\circ g_1\left(\frac{1}{b^{\rho_1}},d^{\rho_2}\right)+\,^{\rho_1}I^{\alpha}_{1/a-}f\circ g_1\left(\frac{1}{b^{\rho_1}},c^{\rho_2}\right)\right]$$
$$+\frac{\rho_2^{\beta+1}\Gamma(\beta+1)}{4}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\left[\,^{\rho_2}I^{\beta}_{1/c-}f\circ g_2\left(a^{\rho_1},\frac{1}{d^{\rho_2}}\right)+\,^{\rho_2}I^{\beta}_{1/c-}f\circ g_2\left(b^{\rho_1},\frac{1}{d^{\rho_2}}\right)\right.$$
$$\left.+\,^{\rho_2}I^{\alpha}_{1/d+}f\circ g_2\left(a^{\rho_1},\frac{1}{c^{\rho_2}}\right)+\,^{\rho_2}I^{\alpha}_{1/d+}f\circ g_2\left(b^{\rho_1},\frac{1}{c^{\rho_2}}\right)\right],$$

and $A_t=t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}$, $B_\lambda=\lambda^{\rho_2}c^{\rho_2}+(1-\lambda^{\rho_2})d^{\rho_2}$*. Also,* $g(x^{\rho_1},y^{\rho_2})=(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}})$, $g_1(x^{\rho_1},y^{\rho_2})=(\frac{1}{x^{\rho_1}},y^{\rho_2})$ *and* $g_2(x^{\rho_1},y^{\rho_2})=(x^{\rho_1},\frac{1}{y^{\rho_2}})$.

Proof. By integration by parts and using the change of variable $x^{\rho_1}=\frac{A_t}{a^{\rho_1}b^{\rho_1}}$ and $y^{\rho_2}=\frac{B_\lambda}{c^{\rho_2}d^{\rho_2}}$, we find that

(11)

$$
\begin{aligned}
I_1 &= \int_0^1 \int_0^1 \frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2 B_\lambda^2} t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) d\lambda dt \\
&= \int_0^1 \frac{\lambda^{\rho_2\beta}}{B_\lambda^2}\lambda^{\rho_2-1}\Bigg\{ \frac{t^{\rho_1\alpha}}{\rho_1 a^{\rho_1}b^{\rho_1}(b^{\rho_1}-a^{\rho_1})}\frac{\partial f}{\partial \lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)\Bigg|_0^1 \\
&- \frac{\alpha}{a^{\rho_1}b^{\rho_1}(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\rho_1\alpha-1}\frac{\partial f}{\partial \lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) dt\Bigg\} d\lambda \\
&= \frac{1}{\rho_1 a^{\rho_1}b^{\rho_1}(b^{\rho_1}-a^{\rho_1})}\int_0^1 \frac{\lambda^{\rho_2\beta}}{B_\lambda^2}\lambda^{\rho_2-1}\frac{\partial f}{\partial \lambda}\left(b^{\rho_1}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) d\lambda \\
&- \frac{\alpha}{a^{\rho_1}b^{\rho_1}(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\rho_1\alpha-1}\left\{\int_0^1 \frac{\lambda^{\rho_2\beta}}{B_\lambda^2}\lambda^{\rho_2-1}\frac{\partial f}{\partial \lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) d\lambda\right\} dt \\
&= \frac{1}{\rho_1\rho_2 a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})} f(b^{\rho_1}, d^{\rho_2}) \\
&- \frac{\beta}{\rho_1 a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\int_0^1 \lambda^{\beta-1} f\left(b^{\rho_1}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) d\lambda \\
&- \frac{\alpha}{\rho_2 a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\int_0^1 t^{\rho_1\alpha-1} f\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, d^{\rho_2}\right) dt \\
&+ \frac{\alpha\beta}{a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\int_0^1\int_0^1 t^{\rho_1\alpha-1}\lambda^{\rho_2\beta-1} f\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) d\lambda dt \\
&= \frac{1}{a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\Bigg[\frac{f(b^{\rho_1}, d^{\rho_2})}{\rho_1\rho_2} \\
&- \frac{\rho_2^\beta\Gamma(\beta+1)}{\rho_1}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^\beta \, {}^{\rho_2}I^\alpha_{1/d+} f\circ g_2\left(b^{\rho_1}, \frac{1}{c^{\rho_2}}\right) \\
&- \frac{\rho_1^\alpha\Gamma(\alpha+1)}{\rho_2}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^\alpha \, {}^{\rho_1}I^\alpha_{1/b+} f\circ g_1\left(\frac{1}{a^{\rho_1}}, d^{\rho_2}\right) + \rho_1^\alpha\rho_2^\beta\Gamma(\alpha+1)\Gamma(\beta+1) \\
&\times \left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^\alpha\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^\beta \, {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+} f\circ g\left(\frac{1}{a^{\rho_1}}, \frac{1}{c^{\rho_2}}\right)\Bigg].
\end{aligned}
$$

Analogously, we have

(12)

$$
\begin{aligned}
I_2 &= \int_0^1\int_0^1 \frac{(1-t^{\rho_1})^\alpha\lambda^{\rho_2\beta}}{A_t^2 B_\lambda^2} t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t}, \frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right) d\lambda dt \\
&= \frac{1}{a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\Bigg[-\frac{f(a^{\rho_1}, c^{\rho_2})}{\rho_1\rho_2}
\end{aligned}
$$

$$+\frac{\rho_2^{\beta}\Gamma(\beta+1)}{\rho_1}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\ {}^{\rho_2}I^{\alpha}_{1/d+}f\circ g_2\left(a^{\rho_1},\frac{1}{c^{\rho_2}}\right)$$
$$+\frac{\rho_1^{\alpha}\Gamma(\alpha+1)}{\rho_2}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\ {}^{\rho_1}I^{\alpha}_{1/a-}f\circ g_1\left(\frac{1}{b^{\rho_1}},d^{\rho_2}\right)-\rho_1^{\alpha}\rho_2^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)$$
$$\left.\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\right],$$

(13)

$$I_3=\int_0^1\int_0^1\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)d\lambda dt$$
$$=\frac{1}{a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\left[-\frac{f(b^{\rho_1},c^{\rho_2})}{\rho_1\rho_2}\right.$$
$$+\frac{\rho_2^{\beta}\Gamma(\beta+1)}{\rho_1}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\ {}^{\rho_2}I^{\alpha}_{1/c-}f\circ g_2\left(b^{\rho_1},\frac{1}{d^{\rho_2}}\right)$$
$$+\frac{\rho_1^{\alpha}\Gamma(\alpha+1)}{\rho_2}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\ {}^{\rho_1}I^{\alpha}_{1/b+}f\circ g_1\left(\frac{1}{a^{\rho_1}},c^{\rho_2}\right)-\rho_1^{\alpha}\rho_2^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)$$
$$\left.\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)\right],$$

and

(14)

$$I_4=\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)d\lambda dt$$
$$=\frac{1}{a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}\left[\frac{f(a^{\rho_1},d^{\rho_2})}{\rho_1\rho_2}\right.$$
$$-\frac{\rho_2^{\beta}\Gamma(\beta+1)}{\rho_1}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\ {}^{\rho_2}I^{\alpha}_{1/c-}f\circ g_2\left(a^{\rho_1},\frac{1}{d^{\rho_2}}\right)$$
$$-\frac{\rho_1^{\alpha}\Gamma(\alpha+1)}{\rho_2}\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\ {}^{\rho_1}I^{\alpha}_{1/a-}f\circ g_1\left(\frac{1}{b^{\rho_1}},c^{\rho_2}\right)+\rho_1^{\alpha}\rho_2^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)$$
$$\left.\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)\right].$$

Then from equalities (11)-(14), we find

(15)

$$
\begin{aligned}
I_1 - I_2 - I_3 + I_4 &= \frac{f(a^{\rho_1}, c^{\rho_2}) + f(a^{\rho_1}, d^{\rho_2}) + f(b^{\rho_1}, c^{\rho_2}) + f(b^{\rho_1}, d^{\rho_2})}{\rho_1 \rho_2 a^{\rho_1} b^{\rho_1} c^{\rho_2} d^{\rho_2} (b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})} \\
&- \frac{\rho_2^{\beta} \Gamma(\beta+1)}{\rho_1 a^{\rho_1} b^{\rho_1} c^{\rho_2} d^{\rho_2} (b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})} \left(\frac{c^{\rho_2} d^{\rho_2}}{d^{\rho_2} - c^{\rho_2}} \right)^{\beta} \\
&\times \left[{}^{\rho_2} I^{\beta}_{1/c-} f \circ g_2 \left(a^{\rho_1}, \frac{1}{d^{\rho_2}} \right) + {}^{\rho_2} I^{\beta}_{1/c-} f \circ g_2 \left(b^{\rho_1}, \frac{1}{d^{\rho_2}} \right) \right. \\
&+ \left. {}^{\rho_2} I^{\alpha}_{1/d+} f \circ g_2 \left(a^{\rho_1}, \frac{1}{c^{\rho_2}} \right) + {}^{\rho_2} I^{\alpha}_{1/d+} f \circ g_2 \left(b^{\rho_1}, \frac{1}{c^{\rho_2}} \right) \right] \\
&- \frac{\rho_1^{\alpha} \Gamma(\alpha+1)}{\rho_2 a^{\rho_1} b^{\rho_1} c^{\rho_2} d^{\rho_2} (b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})} \left(\frac{a^{\rho_1} b^{\rho_1}}{b^{\rho_1} - a^{\rho_1}} \right)^{\alpha} \\
&\times \left[{}^{\rho_1} I^{\alpha}_{1/b+} f \circ g_1 \left(\frac{1}{a^{\rho_1}}, d^{\rho_2} \right) + {}^{\rho_1} I^{\alpha}_{1/b+} f \circ g_1 \left(\frac{1}{a^{\rho_1}}, c^{\rho_2} \right) \right. \\
&+ \left. {}^{\rho_1} I^{\alpha}_{1/a-} f \circ g_1 \left(\frac{1}{b^{\rho_1}}, d^{\rho_2} \right) + {}^{\rho_1} I^{\alpha}_{1/a-} f \circ g_1 \left(\frac{1}{b^{\rho_1}}, c^{\rho_2} \right) \right] \\
&+ \frac{\rho_1^{\alpha} \rho_2^{\beta} \Gamma(\alpha+1)\Gamma(\beta+1)}{a^{\rho_1} b^{\rho_1} c^{\rho_2} d^{\rho_2} (b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})} \left(\frac{a^{\rho_1} b^{\rho_1}}{b^{\rho_1} - a^{\rho_1}} \right)^{\alpha} \left(\frac{c^{\rho_2} d^{\rho_2}}{d^{\rho_2} - c^{\rho_2}} \right)^{\beta} \\
&\times \left[{}^{\rho_1,\rho_2} I^{\alpha,\beta}_{1/a-,1/c-} f \circ g \left(\frac{1}{b^{\rho_1}}, \frac{1}{d^{\rho_2}} \right) + {}^{\rho_1,\rho_2} I^{\alpha,\beta}_{1/a-,1/d+} f \circ g \left(\frac{1}{b^{\rho_1}}, \frac{1}{c^{\rho_2}} \right) \right. \\
&+ \left. {}^{\rho_1,\rho_2} I^{\alpha,\beta}_{1/b+,1/c-} f \circ g \left(\frac{1}{a^{\rho_1}}, \frac{1}{d^{\rho_2}} \right) + {}^{\rho_1,\rho_2} I^{\alpha,\beta}_{1/b+,1/d+} f \circ g \left(\frac{1}{a^{\rho_1}}, \frac{1}{c^{\rho_2}} \right) \right].
\end{aligned}
$$

Multiplying by $\frac{\rho_1 \rho_2 a^{\rho_1} b^{\rho_1} c^{\rho_2} d^{\rho_2} (b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})}{4}$ on both sides of the equality (15), we get the required equality (9). □

Theorem 2.4. *Let* $f : \Delta = [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ *be a partial differentiable mapping on* Δ *with* $0 < a < b$ *and* $0 < c < b$. *If* $|\partial^2 f / \partial t \partial \lambda|$ *is a* *harmonically convex on the co-ordinates on Δ, then following inequality holds:*

(16)

$$\Bigg| \frac{f(a^{\rho_1}, c^{\rho_2}) + f(a^{\rho_1}, d^{\rho_2}) + f(b^{\rho_1}, c^{\rho_2}) + f(b^{\rho_1}, d^{\rho_2})}{4} + \frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}$$

$$\times \left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha} \left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta} \Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-} f\circ g\left(\frac{1}{b^{\rho_1}}, \frac{1}{d^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+} f\circ g\left(\frac{1}{b^{\rho_1}}, \frac{1}{c^{\rho_2}}\right)$$

$$+ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-} f\circ g\left(\frac{1}{a^{\rho_1}}, \frac{1}{d^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+} f\circ g\left(\frac{1}{a^{\rho_1}}, \frac{1}{c^{\rho_2}}\right)\Bigg] - \Xi\Bigg|$$

$$\leq \frac{a^{\rho_1}c^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4b^{\rho_1}d^{\rho_2}(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)} \Bigg[\vartheta_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1}, c^{\rho_2})\right| + \vartheta_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1}, d^{\rho_2})\right|$$

$$+ \vartheta_3\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1}, c^{\rho_2})\right| + \vartheta_4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1}, d^{\rho_2})\right|\Bigg],$$

where

(17)

$$\vartheta_1 = (\alpha+1)(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2, \alpha+2; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, \beta+2; \beta+3; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$

$$+ (\beta+1)\ {}^{\rho_1}_{2}G_1\left(2, 2; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_2}}\right)\ {}^{\rho_2}_{2}G_1\left(2, \beta+2; \beta+3; 1-\frac{c^{\rho_1}}{d^{\rho_1}}\right)$$

$$+ (\alpha+1)\ {}^{\rho_1}_{2}G_1\left(2, \alpha+2; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, 2; \beta+3; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$

$$+ {}^{\rho_1}_{2}G_1\left(2, 2; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, 2; \beta+3; 1-\frac{c^{\rho_1}}{d^{\rho_1}}\right),$$

(18)

$$\vartheta_2 = (\beta+1)\ {}^{\rho_1}_{2}G_1\left(2, \alpha+1; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, \beta+2; \beta+3; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$

$$+ (\alpha+1)(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2, 1; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, \beta+2; \beta+3; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$

$$+ {}^{\rho_1}_{2}G_1\left(2, \alpha+1; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, 2; \beta+3; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$

$$+ (\alpha+1)\ {}^{\rho_1}_{2}G_1\left(2, 1; \alpha+3; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2, 2; \beta+3; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

(19)

$$\vartheta_3 = (\alpha+1)\ {}^{\rho_1}_{2}G_1\left(2,\alpha+2;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$
$$+\ {}^{\rho_1}_{2}G_1\left(2,2;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$
$$+(\alpha+1)(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,\alpha+2;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$
$$+(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,2;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

(20)

$$\vartheta_4 =\ {}^{\rho_1}_{2}G_1\left(2,\alpha+1;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$
$$+(\alpha+1)\ {}^{\rho_1}_{2}G_1\left(2,1;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$
$$+(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,\alpha+1;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)$$
$$+(\alpha+1)(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,1;\alpha+3;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,1;\beta+3;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right)\Bigg).$$

Proof. Using Lemma 2.3 and triangular inequality, we have

(21)

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}$$
$$\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)$$
$$+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)$$
$$+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\Bigg]-\Xi\Bigg|$$

$$\leq \frac{\rho_1\rho_2 a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_1}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4}\Bigg[\int_0^1\int_0^1 \frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}$$

$$\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)\right|d\lambda dt-\int_0^1\int_0^1 \frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}$$

$$\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)\right|d\lambda dt-\int_0^1\int_0^1 \frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}$$

$$\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)\right|d\lambda dt+\int_0^1\int_0^1 \frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}t^{\rho_1-1}\lambda^{\rho_2-1}$$

$$\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)\right|d\lambda dt\Bigg].$$

Now using co-ordinated harmonically convexity of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, we get

(22)

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}$$

$$\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\Bigg[\,{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+\,{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)$$

$$+\,{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+\,{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\Bigg]-\Xi\Bigg|$$

$$\leq \frac{\rho_1\rho_2 a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_1}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4}\Bigg[\int_0^1\int_0^1\Bigg[\frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}$$

$$+\frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}+\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}\Bigg]t^{\rho_1-1}\lambda^{\rho_2-1}$$

$$\times\Bigg\{t^{\rho_1}\lambda^{\rho_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},c^{\rho_2})\right|+(1-t^{\rho_1})\lambda^{\rho_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},c^{\rho_2})\right|$$

$$+t^{\rho_1}(1-\lambda^{\rho_2})\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},d^{\rho_2})\right|+(1-t^{\rho_1})(1-\lambda^{\rho_2})\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},d^{\rho_2})\right|\Bigg\}d\lambda dt\Bigg]$$

$$
\begin{aligned}
&= \frac{\rho_1\rho_2 a^{\rho_1} b^{\rho_1} c^{\rho_2} d^{\rho_1}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4}\Bigg[\int_0^1\int_0^1 t^{\rho_1}\lambda^{\rho_2}\Big\{\frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}\\
&+\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}\Big\}t^{\rho_1-1}\lambda^{\rho_2-1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},c^{\rho_2})\right|d\lambda dt\\
&+\int_0^1\int_0^1(1-t^{\rho_1})\lambda^{\rho_2}\Big\{\frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}\\
&+\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}\Big\}t^{\rho_1-1}\lambda^{\rho_2-1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},c^{\rho_2})\right|d\lambda dt\\
&+\int_0^1\int_0^1 t^{\rho_1}(1-\lambda^{\rho_2})\Big\{\frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}\\
&+\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}\Big\}t^{\rho_1-1}\lambda^{\rho_2-1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},d^{\rho_2})\right|d\lambda dt\\
&+\int_0^1\int_0^1(1-t^{\rho_1})(1-\lambda^{\rho_2})\Big\{\frac{t^{\rho_1\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}\lambda^{\rho_2\beta}}{A_t^2B_\lambda^2}\\
&+\frac{t^{\rho_1\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}+\frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_t^2B_\lambda^2}\Big\}t^{\rho_1-1}\lambda^{\rho_2-1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},d^{\rho_2})\right|d\lambda dt\Bigg].
\end{aligned}
$$

After some calculations we get the desired result. □

Theorem 2.5. *Let $f:\Delta=[a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0<a<b$ and $0<c<b$. If $|\partial^2 f/\partial t\partial\lambda|^q$, $q>1$, is a harmonically convex on the co-ordinates on Δ, then following holds:*

(23)

$$
\begin{aligned}
&\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\\
&\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\Bigg[\,{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\\
&+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\Bigg]-\Xi\Bigg|\\
&\le\frac{\rho_1\rho_2a^{\rho_1}c^{\rho_2}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4b^{\rho_1}d^{\rho_2}[\rho_1\rho_2(p\alpha+1)(p\beta+1)]^{1/p}}\left[\psi_1^{1/p}+\psi_2^{1/p}+\psi_3^{1/p}+\psi_4^{1/p}\right]\\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},c^{\rho_2})\right|^q+\rho_1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},d^{\rho_2})\right|^q+\rho_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},c^{\rho_2})\right|^q+\rho_1\rho_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},d^{\rho_2})\right|^q}{(\rho_1+1)(\rho_2+1)}\right)^{1/q},
\end{aligned}
$$

where

$$(24)\quad \psi_1 = {}^{\rho_1}_{2}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right) {}^{\rho_2}_{2}G_1\left(2p, p\beta+1; p\beta+2; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$(25)\quad \psi_2 = {}^{\rho_1}_{2}G_1\left(2p, 1; p\alpha+2; 1-\frac{a^{\rho_1}}{b^{\rho_2}}\right) {}^{\rho_2}_{2}G_1\left(2p, p\beta+1; p\beta+2; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$(26)\quad \psi_3 = {}^{\rho_1}_{2}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right) {}^{\rho_2}_{2}G_1\left(2p, 1; p\beta+2; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$(27)\quad \psi_4 = {}^{\rho_1}_{2}G_1\left(2p, 1; p\alpha+2; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right) {}^{\rho_2}_{2}G_1\left(2p, 1; p\beta+2; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right).$$

Proof. Applying the Holder's inequality for double integrals in (21), we get

(28)

$$\begin{aligned}
&\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\\
&\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\\
&+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\Bigg]-\Xi\Bigg|\\
&\le\frac{\rho_1\rho_2a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_1}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4}\Bigg[\left(\int_0^1\int_0^1\frac{t^{p\rho_1\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}\\
&+\left(\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{p\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}+\Bigg(\int_0^1\int_0^1\frac{t^{p\rho_1\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\\
&\times\lambda^{p(\rho_2-1)}d\lambda dt\Bigg)^{1/p}+\left(\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{p\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}\Bigg]\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(\frac{a^{\rho_1}b^{\rho_1}}{A_t},\frac{c^{\rho_2}d^{\rho_2}}{B_\lambda}\right)\right|^q d\lambda dt\right)^{1/q}.
\end{aligned}$$

Using co-ordinated harmonically convexity of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, we get

(29)

$$\left|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\right.$$

$$\times\left(\frac{a^{\rho_1}b^{\rho_1}}{b^{\rho_1}-a^{\rho_1}}\right)^{\alpha}\left(\frac{c^{\rho_2}d^{\rho_2}}{d^{\rho_2}-c^{\rho_2}}\right)^{\beta}\left[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/c-}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/a-,1/d+}f\circ g\left(\frac{1}{b^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\right.$$

$$\left.\left.+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/c-}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{d^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/b+,1/d+}f\circ g\left(\frac{1}{a^{\rho_1}},\frac{1}{c^{\rho_2}}\right)\right]-\Xi\right|$$

$$\leq\frac{\rho_1\rho_2a^{\rho_1}b^{\rho_1}c^{\rho_2}d^{\rho_1}(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_1})}{4}\left[\left(\int_0^1\int_0^1\frac{t^{p\rho_1\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}\right.$$

$$+\left(\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{p\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}+\left(\int_0^1\int_0^1\frac{t^{p\rho_1\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\right.$$

$$\left.\left.\times\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}+\left(\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{p\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt\right)^{1/p}\right]$$

$$\times\left(\int_0^1\int_0^1\left\{t^{\rho_1}\lambda^{\rho_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},c^{\rho_2})\right|^q+(1-t^{\rho_1})\lambda^{\rho_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},c^{\rho_2})\right|^q\right.\right.$$

$$\left.\left.+t^{\rho_1}(1-\lambda^{\rho_2})\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a^{\rho_1},d^{\rho_2})\right|^q+(1-t^{\rho_1})(1-\lambda^{\rho_2})\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b^{\rho_1},d^{\rho_2})\right|^q\right\}d\lambda dt\right)^{1/q}.$$

Where

$$\int_0^1\int_0^1\frac{t^{p\rho_1\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt$$
$$\leq\int_0^1\int_0^1\frac{t^{p\rho_1\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{\rho_1-1}\lambda^{\rho_2-1}d\lambda dt=\frac{b^{-2p}d^{-2p}}{\rho_1\rho2(p\alpha+1)(p\beta+1)}\times$$
$${}^{\rho_1}_{2}G_1\left(2p,p\alpha+1;p\alpha+2;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2p,p\beta+1;p\beta+2;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{p\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt$$
$$\leq\int_0^1\int_0^1\frac{(1-t^{\rho_1})^{p\alpha}\lambda^{p\rho_2\beta}}{A_t^{2p}B_\lambda^{2p}}t^{\rho_1-1}\lambda^{\rho_2-1}d\lambda dt=\frac{b^{-2p}d^{-2p}}{\rho_1\rho2(p\alpha+1)(p\beta+1)}\times$$
$${}^{\rho_1}_{2}G_1\left(2p,1;p\alpha+2;1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2p,p\beta+1;p\beta+2;1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$\int_0^1\int_0^1 \frac{t^{p\rho_1\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}} t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt$$
$$\leq \int_0^1\int_0^1 \frac{t^{p\rho_1\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}} t^{\rho_1-1}\lambda^{\rho_2-1}d\lambda dt = \frac{b^{-2p}d^{-2p}}{\rho_1\rho 2(p\alpha+1)(p\beta+1)}\times$$
$${}_2^{\rho_1}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2p, 1; p\beta+2; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$\int_0^1\int_0^1 \frac{(1-t^{\rho_1})^{p\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}} t^{p(\rho_1-1)}\lambda^{p(\rho_2-1)}d\lambda dt$$
$$\leq \int_0^1\int_0^1 \frac{(1-t^{\rho_1})^{p\alpha}(1-\lambda^{\rho_2})^{p\beta}}{A_t^{2p}B_\lambda^{2p}} t^{\rho_1-1}\lambda^{\rho_2-1}d\lambda dt = \frac{b^{-2p}d^{-2p}}{\rho_1\rho 2(p\alpha+1)(p\beta+1)}\times$$
$${}_2^{\rho_1}G_1\left(2p, 1; p\alpha+2; 1-\frac{a^{\rho_1}}{b^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2p, 1; p\beta+2; 1-\frac{c^{\rho_2}}{d^{\rho_2}}\right),$$

$$\int_0^1\int_0^1 t^{\rho_1}\lambda^{\rho_2}d\lambda dt = \frac{1}{(\rho_1+1)(\rho_2+1)},$$

$$\int_0^1\int_0^1 (1-t^{\rho_1})\lambda^{\rho_2}d\lambda dt = \frac{\rho_1}{(\rho_1+1)(\rho_2+1)},$$

$$\int_0^1\int_0^1 t^{\rho_1}(1-\lambda^{\rho_2})d\lambda dt = \frac{\rho_2}{(\rho_1+1)(\rho_2+1)},$$

$$\int_0^1\int_0^1 (1-t^{\rho_1})(1-\lambda^{\rho_2})d\lambda dt = \frac{\rho_1\rho_2}{(\rho_1+1)(\rho_2+1)}.$$

Putting the values of above integrals in (29) and after some simplification, we get the required result (23). □

Remark 2.6. *By taking $\rho_1 = \rho_2 = 1$ in Theorem 2.1, Lemma 2.3, Theorem 2.4 and in Theorem 2.5, we get similar results for co-ordinated harmonically convex functions via Riemann-Liouville fractional integrals.*

Conclusion: From Theorem 2.1 we get Hermite-Hadamard inequality for co-ordinated harmonically convex on a rectangle via Katugampola fractional integrals. From identity proved in Lemma 2.3 we get some more Hermite-Hadamard type inequalities for co-ordinated harmonically convex on a rectangle via Katugampola fractional integrals.

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