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Optimal ordering policies using a discounted cash-flow analysis when stock – dependent demand and a trade credit is linked to order quantity

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Abstract

In this article, the purchaser's optimal pricing and ordering policy is developed when units in inventory are subject to deterioration at a constant rate. The demand is assumed to be stock dependent and the supplier offers to the purchaser different trade credits. We solve the inventory problem by using a discounted cash-flow (DCF) approach, characterize the optimal solution, and obtain some theoretical results to find the optimal order quantity and the optimal replenishment time. Numerical examples are given to illustrate the proposed model. Sensitivity analysis for stock dependent parameter and deterioration rate is carried out to derive managerial in signets.

Keywords: Inventory, deterioration, stock dependent demand, discounted cash – flow, trade credit.

1 Introduction

In the traditional EOQ model, it is assumed that the purchaser must pay to the supplier for the items as soon as the items are received. However, in today's business transactions, it is more and more common to see that the supplier provides a permissible delay in payments or a price discount to the purchaser if the order quantity is greater than or equal to a predetermined quantity. To the supplier, a permissible delay in payments or a price discount is an efficient method to stimulate more sales. In practice, the supplier is willing to offer the retailer a certain credit period without interest during the permissible delay period to promote market competition. Before the end of the trade credit period, the retailer can sell the goods and accumulate revenue and earn interest. A higher interest is charged if the payment is not settled by the end of the trade credit period. Conversely, to the purchaser, the supplier's trade credit reduces his/her purchase cost. Goyal (1985) used the average cost approach to establish an EOQ model and analyze the effect of trade credit on the optimal inventory policy. Aggarwal and Jaggi (1995) extended Goyal's (1985) model to allow for an exponential deterioration rate under the condition of permissible delay in payments. Jamal, Sarker, and Wang (1997) then further extended the model to allow for shortages. More related articles can be found in the work by Arcelus, Shah, and Srinivasan (2001), Jaber (2007), Sana and Chaudhuri (2008), etc. All these models assume that suppliers offer a delayed payment period to retailers, but the retailers fail to offer the delayed payment to its customers. Chung and Huang (2007) continued to amend Huang (2006) to propose an EOQ model for deteriorating items under the two-level trade credit policy. Jaggi, Goyal, and Goel (2008) established an EOQ model under a two-level trade credit policy with credit linked demand. In contrast, Teng and Chang (2009) modified Huang (2007) by relaxing the assumption that the trade credit period offered by the supplier is longer than the trade credit period offered by the retailer. It is worthwhile noting that all these models discussed optimal inventory strategy under two-level trade credit from the perspective of the buyer or the supplier only. Subsequently, several models concerning the integrated inventory model with trade credit policies have continued to be proposed by Sarker, Jamal, and Wang (2000), Jaber and Osman (2006), Yang and Wee (2006), Chen and Kang (2007), Su, Ouyang, Ho, and Chang (2007), and Ho, Ouyang, and Su (2008).Su et al. (2007) assumed that the supplier and the retailer adopted a two-level trade credit policy. However, Subsequently, several models concerning the integrated inventory model with trade credit policies have continued to be proposed by Sarker, Jamal, and Wang (2000), Jaber and Osman (2006), Yang and Wee (2006), Chen and Kang (2007), Su, Ouyang, Ho, and Chang (2007), and Ho, Ouyang, and Su (2008).Suet al. (2007) assumed that the retailer obtained a longer trade credit period from the supplier and provided a relatively shorter trade credit period to customers. Additionally, the demand rate only depends on the length of a customer's credit period (i.e. credit-linked demand rate). Ouyang, Chang, and Teng (2005) generalized Goyal's (1985) model to obtain an optimal ordering policy for the retailer when the supplier provides not only a cash discount but also a permissible delay in payments. Teng, Ouyang, and Chen (2006) developed an economic production quantity model in which the manufacturer receives the supplier trade credit and provides the customer trade credit simultaneously. In 2007, Ouyang, Wu, and Yang proposed an economic order quantity inventory model with limited storage capacity. They considered the situation when he supplier provides a cash discount and a permissible delay in payments for the retailer. Ho, Ouyang, and Su (2008) established an integrated supplier-buyer inventory model with the assumption that the market demand is sensitive to the retail price and discussed the trade credit policy including a cash discount and delayed payment. Chang, Ho, Ouyang, and Su (2009) incorporated the concept of vendor-buyer integration and order-size-dependent trade credit. They presented a stylized model to determine the optimal strategy for an integrated vendor-buyer inventory system under the condition of trade credit linked to the order quantity. Many related papers can be found in Hwang and Shinn (1997), Jamal, Sarker, and Wang (2000), Liao, Tsai, and Su (2000), Teng (2002), Abad and Jaggi (2003), Arcelus, Shah, and Srinivasan (2003), Chang, Ouyang, and Teng (2003), Shinn and Hwang (2003), Chang (2004), Chung and Liao (2004), Teng (2006), Teng, Chang, Chern, and Chan (2007), Teng and Goyal (2007) and others.

In the above papers, the researchers adopted the average cost approach, and did not consider the effect of the time-value of it is necessary to take the effect of the time-value of money on the inventory policy into consideration. Trippi and Lewin (1974) and Kim and Chung (1990) recognized the need to explore inventory problems by using DCF approach or the present value concept. Chung (1989) adopted the DCF approach for the analysis of the optimal inventory policy in the presence of a trade credit. Wee and Law (2001) developed a deteriorating inventory model with price-dependent demand and took into account the time-value of money. They applied the DCF approach for problem analysis. Grubbström and Kingsman (2004) discussed the problem of determining the optimal ordering quantities of a purchased item where there are step changes in price and the net present value (NPV) principle is applied. Wee, Yu Jonas, and Law (2005) established a two warehouse inventory model with partial backordering and Weibull distribution deterioration, in which the DCF and optimization framework are presented to derive the replenishment policy that minimizes the total present value cost per unit time. Jaggi, Aggarwal, and Goel (2006) presented the optimal inventory replenishment policy for deteriorating items under inflationary conditions by using the DCF approach over a finite planning horizon. Dye, Ouyang, and Hsieh (2007) considered an infinite horizon, single product economic order quantity where demand and he deterioration rate are continuous and differentiable functions of price and time, respectively. They applied the DCF approach to deal with the problem. Hsieh, Dye, and Ouyang (2008) established an inventory model for deteriorating items with two levels of storages, permitting shortage and complete backlogging and used the NPV of total cost as the objective function for the generalized inventory system. Several interesting and relevant papers related to trade credits using the DCF approach are Chapman, Ward, Cooper, and Page (1984), Daellenbach (1986), Ward and Chapman (1987), Jaggi and Aggarwal (1994), Chung and Huang (2000), Chung and Liao (2006), Soni, Gor, and Shah (2006), Teng (2006) and others.

In this paper, we use the DCF approach to establish an inventory model for deteriorating items with trade credit based on the order quantity. The supplier offers his/her customers two alternatives linked to order quantity. One is that if the order quantity is greater than or equal to a predetermined quantity, then the supplier provides a longer permissible delay in payments. Otherwise, the total purchase cost must be paid by a fixed credit period. The other is that if the order quantity is greater than or equal to a predetermined quantity, then there is a quantity discount on price and the balance must be paid by a fixed credit period. Otherwise, there is no price discount and the total purchase cost must be paid by a fixed credit period. We then study the necessary and sufficient conditions for finding the optimal solution to the problem. Finally, some numerical examples are presented to illustrate the theoretical results. This article is organized as follows. In Section 2, we provide the notation and assumptions. In Section 3, we established the discount cash-flow models. Section 4 contains the theoretical results. We obtain a theorem to show the optimal solution not only exists but also is unique for the present value functions of all future cash outflows. The comparison among different policies is in Section 5. Section 6 provides a discussion of the different policy alternatives based on different parameter values. Several numerical examples are given to illustrate the results in this paper in Section 7. Finally, Section 8 draws conclusions and suggests potential directions for future research.

2 Assumptions and Notations:

The mathematical models in this research article are developed using following assumptions:

1. The demand rate for the item is stock dependent.
2. Shortages are not allowed.
3. The replenishment rate is instantaneous.
4. Time horizon is infinite.
5. The units in inventory deteriorate at a constant rate 'θ', $0 \le \theta \le 1$. The deteriorated units can neither be repaired nor replaced during the cycle time.
6. The supplier offers his/her customers two alternatives linked to order quantity: (a) A permissible delay. If the order quantity is greater than or equal to a predetermined quantity Q_{pd}, then the total purchase cost must be paid by N. Otherwise, the total purchase cost must be paid by M (with N > M).
   (b) A price discount. If the order quantity is greater than or equal to a predetermined quantity Q_{cd}, then there is a quantity discount on price and the balance must be paid by M. Otherwise, there is no price discount and the total purchase cost must be paid by M.

In addition, the following notations are used throughout this paper:

i : the out-of-pocket holding cost as a proportion of the item value per unit time
C : the unit purchasing cost.
A : the ordering cost per order.
r : the opportunity cost (i.e. the continuous discounting rate) per unit time.
Q : the order quantity
Q_{pd} : the minimum order quantity at which the permissible delay in payments M_2 is permitted
T_{pd} : the time interval such that Q_{pd} units are depleted to zero due to both demand and deterioration
Q_{cd} : the minimum order quantity at which the cash discount is available
T_{cd} : the time interval such that Q_{cd} units are depleted to zero due to both demand and deterioration
d : the cash discount rate, $0 < d < 1$
T : the cycle time (a decision variable).
$I(t)$: the inventory level at any instant of time t, $0 \le t \le T$
M : the credit period at which the order quantity is less than Q_{pd} (or Q_{cd})
N : the credit period at which the order quantity is greater than or equal to Q_{pd}, with $T_{pd} > N > M$
θ : constant rate of deterioration, $0 \le \theta \le 1$
$R(I(t))$: the demand rate at time t. we consider $R(I(t)) = \alpha + \beta I(t)$ where $\alpha > 0$ is constant demand and $0 \le \beta < 1$ denotes rate of change of demand due to stock.
PV(T) : the present value of cash flows for the first replenishment cycle.
APV(T) : the present value of all future cash flows.
Q* : the optimal order quantity.
T* : the optimal replenishment time interval.
APV* : the minimum present value of all future cash flows, i.e. APV* = APV(T*)

3 Discounted cash – flow mathematical Models:

The depletion of the inventory is due to stock – dependent demand and deterioration of units. The inventory level at any instant of is governed by the differential equation:

$$\frac{dI(t)}{dt}+\theta I(t)=-R(I(t)),\quad 0\le t\le T \tag{1}$$

with the initial condition I(0) = Q and boundary condition I(T) = 0The solution of Eq. (1) is given by,

$$I(t)=\frac{\alpha}{(\beta+\theta)}\left[e^{(\beta+\theta)(T-t)}-1\right],\quad 0\le t\le T \tag{2}$$

and order quantity is

$$Q=I(0)=\frac{\alpha}{(\beta+\theta)}\left[e^{(\beta+\theta)T}-1\right] \tag{3}$$

From (3), we can obtain the time interval such that Q_{pd} units are depleted to zero due to both stock dependent demand and deterioration as

$$T_{pd}=\frac{1}{(\beta+\theta)}\ln\left[1+\frac{(\beta+\theta)Q_{pd}}{\alpha}\right] \tag{4}$$

Consequently, it is easy to see that the inequality Q < Q_{pd} holds if and only if T < T_{pd}.

Similarly,

$$T_{cd}=\frac{1}{(\beta+\theta)}\ln\left[1+\frac{(\beta+\theta)Q_{cd}}{\alpha}\right] \tag{5}$$

and the inequality Q < Q_{cd} holds if and only if T < T_{cd}.

Based on the values of T, T_{pd} and T_{cd}, there are four cases to be considered: (1) T < T_{pd}, (2) T ≥ T_{pd}, (3) T < T_{cd} and (4) T ≥ T_{cd}. For each case, the corresponding payment time and cash discount rate are presented in Table 1.

Table 1. The payment time and cash discount rate for different cases.

Case	Payment time	Cash discount rate
(1) T < T_{pd}	M	0
(2)T ≥ T_{pd}	N	0
(3) T < T_{cd}	M	0
(4) T ≥ T_{cd}	M	d

Now, we develop the discounted cash-flow model for each case.

Case 1. T < T_{pd} (i.e. the order quantity is less than Q_{pd})

The present value of cash flows for the first cycle; the components are evaluated as following:

1. At the beginning of each replenishment cycle, cash out flows of the ordering cost, A
2. At the end of the credit period, M, the customer pays the full purchase cost, CQ, if the order quantity is less than Q_{pd}. Using (3), $CQ=C\frac{\alpha}{(\beta+\theta)}\left[e^{(\beta+\theta)T_1}-1\right]$; where T_1 stands for the replenishment time interval for case 1.
3. The present value of the purchase cost at the continuous discounting rate r is $CQ=C\frac{\alpha}{(\beta+\theta)}\left[e^{(\beta+\theta)T_1}-1\right]e^{-rM}$

4. The out – of – pocket inventory holding cost at time t is $iCI(t) = \frac{iC\alpha}{(\beta+\theta)}\left[e^{(\beta+\theta)(T_1-t)} - 1\right]$

5. The present value of the out – of – pocket holding cost in a replenishment cycle with the continuous discounting rate r is $\int_0^{T_1}\left[\frac{iC\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)(T_1-t)} - 1\right)\right]e^{-rt}\,dt$

Consequently, the present value of cash flows for the first cycle is

$$PV_1(T_1) = A + \frac{C\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)T_1} - 1\right)e^{-rM} + \frac{iC\alpha}{(\beta+\theta)}\int_0^{T_1}\left[\left(e^{(\beta+\theta)(T_1-t)} - 1\right)\right]e^{-rt}\,dt$$

$$PV_1(T_1) = A + \frac{C\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)T_1} - 1\right)e^{-rM} + \frac{iC\alpha}{(\beta+\theta)}\left[\frac{e^{(\beta+\theta)T_1}}{(\beta+\theta+r)} + \frac{(\beta+\theta)}{r(\beta+\theta+r)}e^{-rT_1} - \frac{1}{r}\right] \tag{6}$$

Then the present value of all future cash flows is

$$APV_1(T_1) = \sum_{n=0}^{\infty} PV_1(T_1)e^{-nrT_1} = PV_1(T_1)\sum_{n=0}^{\infty} e^{-nrT_1} = \frac{PV_1(T_1)}{\left(1-e^{-rT_1}\right)} \tag{7}$$

If M=0 and β = 0, then Eq. (7) is the same as Eq. (6) in Chung and Liao (2006). If β = 0, then Eq. (7) is as that of Chung et. al. (2010)

Case 2. $T \geq T_{pd}$ (i.e. the order quantity is greater than or equal to Q_{pd})

In this case, the supplier offers to customer a permissible delay N in payments. By the similar procedure as described in case 1, we have that the present value of all future cash flows is

$$APV_2(T_2) = \sum_{n=0}^{\infty} PV_2(T_2)e^{-nrT_2} = PV_2(T_2)\sum_{n=0}^{\infty} e^{-nrT_2} = \frac{PV_2(T_2)}{\left(1-e^{-rT_2}\right)} \tag{8}$$

where T_2 is the replenishment cycle length for case 2 and

$$PV_2(T_2) = A + \frac{C\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)T_2} - 1\right)e^{-rN} + \frac{iC\alpha}{(\beta+\theta)}\left[\frac{e^{(\beta+\theta)T_2}}{(\beta+\theta+r)} + \frac{(\beta+\theta)}{r(\beta+\theta+r)}e^{-rT_2} - \frac{1}{r}\right]$$

Case 3. $T < T_{cd}$ (i.e. the order quantity is less than Q_{cd})

In this case, the supplier offers to customer a permissible delay M in payments. Thus the present value of all future cash flows is

$$APV_3(T_3) = \sum_{n=0}^{\infty} PV_3(T_3)e^{-nrT_3} = PV_3(T_3)\sum_{n=0}^{\infty} e^{-nrT_3} = \frac{PV_3(T_3)}{\left(1-e^{-rT_3}\right)} \tag{9}$$

where T_3 is the replenishment cycle length for case 3 and

$$PV_3(T_3) = A + \frac{C\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)T_3} - 1\right)e^{-rM} + \frac{iC\alpha}{(\beta+\theta)}\left[\frac{e^{(\beta+\theta)T_3}}{(\beta+\theta+r)} + \frac{(\beta+\theta)}{r(\beta+\theta+r)}e^{-rT_3} - \frac{1}{r}\right]$$

Case 4. $T \geq T_{cd}$ (i.e. the order quantity is greater than or equal to Q_{cd})

In this case, the supplier offers a permissible delay M in payments also provides a cash discount with the rate d. Thus the present value of all future cash flows is

$$APV_4(T_4) = \sum_{n=0}^{\infty} PV_4(T_4)e^{-nrT_4} = PV_4(T_4)\sum_{n=0}^{\infty} e^{-nrT_4} = \frac{PV_4(T_4)}{\left(1-e^{-rT_4}\right)} \tag{10}$$

where T_4 is the replenishment cycle length for case 4 and

$$PV_4(T_4) = A + \frac{C(1-d)\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)T_4} - 1\right)e^{-rM} + \frac{iC(1-d)\alpha}{(\beta+\theta)}\left[\frac{e^{(\beta+\theta)T_4}}{(\beta+\theta+r)} + \frac{(\beta+\theta)}{r(\beta+\theta+r)}e^{-rT_4} - \frac{1}{r}\right]$$

Combining the Eq. (7) - (10), the present value of all future cash flows for each case is shown as follows:

$$APV_j\left(T_j\right)=\frac{1}{\left(1-e^{-rT_j}\right)}\left\{A+\frac{C(1-d_j)\alpha}{(\beta+\theta)}\left(e^{(\beta+\theta)T_j}-1\right)e^{-rM_j}+\frac{iC(1-d_j)\alpha}{(\beta+\theta)}\left[\frac{e^{(\beta+\theta)T_j}}{(\beta+\theta+r)}+\frac{(\beta+\theta)}{r(\beta+\theta+r)}e^{-rT_j}-\frac{1}{r}\right]\right\} \tag{11}$$

for j = 1, 2, 3, 4, where $M_1 = M_3 = M_4 = M$ and $M_2 = N$, also $d_1 = d_2 = d_3 = 0$ and $d_4 = d$.

Now, to determine the optimal solution that provides the smallest present value of all future cash flows for each case to purchaser.

4. Theoretical results

In this section, we determine the optimal value of T_j (denoted by T_j^*) for case j such that $APV_j\left(T_j^*\right)$ has a minimum value, j = 1, 2, 3, 4. By taking the first derivative of $APV_j\left(T_j\right)$ in (11) with respect to T_j, we have,

$$\frac{dAPV_j\left(T_j\right)}{dT_j}=\frac{1}{\left(1-e^{-rT_j}\right)^2}\left\{-\left(\delta_{1j}+\delta_{2j}e^{(\beta+\theta)T_j}\right)re^{-rT_j}+\left(1-e^{-rT_j}\right)\delta_{2j}(\beta+\theta)e^{(\beta+\theta)T_j}-\frac{iC(1-d_j)\alpha}{(\beta+\theta+r)}e^{-rT_j}\right\} \tag{12}$$

where

$$\delta_{1j}=A-\frac{C(1-d_j)\alpha}{\left(\beta+\theta\right)}e^{-rM_j}-\frac{iC(1-d_j)\alpha}{r\left(\beta+\theta\right)}$$

and

$$\delta_{2j}=\frac{C(1-d_j)\alpha}{\left(\beta+\theta\right)}e^{-rM_j}-\frac{iC(1-d_j)\alpha}{\left(\beta+\theta\right)\left(\beta+\theta+r\right)}$$

Letting $dAPV_j\left(T_j\right)/dT_j=0$, and rearranging terms, we obtain

$$\left(\beta+\theta\right)\delta_{2j}\,e^{(\beta+\theta+r)T_j}-\left(\beta+\theta+r\right)\delta_{2j}\,e^{(\beta+\theta)T_j}=\delta_{1j}\,r+\frac{iC(1-d_j)\alpha}{\left(\beta+\theta+r\right)} \tag{13}$$

From (13), we can obtain the following result.

Theorem 1

(a) The solution $T_j^*\in\left(0,\infty\right)$ which satisfies Eq. (13) not only exists but also is unique, for j = 1, 2, 3, 4.

(b) The solution T_j to (13) is the unique optimal value that minimizes $APV_j\left(T_j\right)$, for j =1, 2, 3, 4

Proof. See Appendix for the detailed proof.

Now, we want to derive the explicit closed-form solution of T_j^*. Utilizing the fact that

$e^{(\beta+\theta+r)T}=1+\left(\beta+\theta+r\right)T+\left[(\beta+\theta+r)T\right]^2/2$. as $(\beta + \theta + r)T$ is small.

$e^{(\beta+\theta)T}=1+\left(\beta+\theta\right)T+\left[(\beta+\theta)T\right]^2/2$. as $(\beta + \theta)T$ is small.

and (13), we obtain

$$\frac{(\beta+\theta)\,r\,\delta_{2j}\,(\beta+\theta+r)T_j^2}{2}=\left(\delta_{1j}+\delta_{2j}\right)r+\frac{iC(1-d_j)\alpha}{\left(\beta+\theta+r\right)}=Ar \tag{14}$$

Consequently, we have the optimal replenishment cycle time

$$T_j^*\approx\sqrt{\frac{2A}{\left(\beta+\theta\right)\left(\beta+\theta+r\right)\delta_{2j}}}=\sqrt{\frac{2A}{C\left(1-d_j\right)\alpha\left[(\beta+\theta+r)e^{-rM_j}+i\right]}} \tag{15}$$

Letting $M_1 = M$ and $d_1 = 0$, we obtain the optimal replenishment cycle length for Case 1 as approximately equal to

$$T_1^*\approx\sqrt{\frac{2A}{C\alpha\left[(\beta+\theta+r)e^{-rM}+i\right]}} \tag{16}$$

Obviously, $T_1^* < T_{pd}$, we substitute (16) into inequality $T_1^* < T_{pd}$, and obtain that

if and only if $2A < \alpha C\left[(\beta+\theta+r)e^{-rM}+i\right]T_{pd}^2$ (17)

The optimal present value of all future cash flows can be obtained as

$$APV_1\left(T_1^*\right) \approx \frac{1}{\left(1-e^{-rT_1^*}\right)}\left\{A+\frac{1}{(\beta+\theta+r)}\left[\sqrt{2AC\alpha\left[(\beta+\theta+r)e^{-rM}+i\right]}+\theta\right]\right\} - \frac{iC\alpha}{r\left(\beta+\theta+r\right)} \tag{18}$$

Similarly, letting M_2= N and d_2 = 0, we obtain the optimal replenishment cycle length for Case 2 as approximately equal to

$$T_2^* \approx \sqrt{\frac{2A}{C\alpha\left[(\beta+\theta+r)e^{-rN}+i\right]}} \tag{19}$$

Substitute (19) into inequality $T_2^* \geq T_{pd}$, we obtain that

if and only if $2A \geq \alpha C\left[(\beta+\theta+r)e^{-rN}+i\right]T_{pd}^2$ (20)

The optimal present value of all future cash flows can be obtained as

$$APV_2\left(T_2^*\right) \approx \frac{1}{\left(1-e^{-rT_2^*}\right)}\left\{A+\frac{1}{(\beta+\theta+r)}\left[\sqrt{2AC\alpha\left[(\beta+\theta+r)e^{-rN}+i\right]}+\theta\right]\right\} - \frac{iC\alpha}{r\left(\beta+\theta+r\right)} \tag{21}$$

Likewise, let M_3= M and d_3 = 0, we obtain the optimal replenishment cycle length for Case 3 as approximately equal to

$$T_3^* \approx \sqrt{\frac{2A}{C\alpha\left[(\beta+\theta+r)e^{-rM}+i\right]}} \tag{22}$$

Substitute (22) into inequality $T_3^* < T_{cd}$, we obtain that

if and only if $2A < \alpha C\left[(\beta+\theta+r)e^{-rM}+i\right]T_{cd}^2$ (23)

The optimal present value of all future cash flows can be obtained as

$$APV_3\left(T_3^*\right) \approx \frac{1}{\left(1-e^{-rT_3^*}\right)}\left\{A+\frac{1}{(\beta+\theta+r)}\left[\sqrt{2AC\alpha\left[(\beta+\theta+r)e^{-rM}+i\right]}+\theta\right]\right\} - \frac{iC\alpha}{r\left(\beta+\theta+r\right)} \tag{24}$$

Last, letting M_4= M and d_4 = d, we can obtain the optimal replenishment cycle length for Case 4 as approximately equal to

$$T_4^* \approx \sqrt{\frac{2A}{C\alpha(1-d)\left[(\beta+\theta+r)e^{-rM}+i\right]}} \tag{25}$$

Substitute (25) into inequality $T_4^* \geq T_{cd}$, we obtain that

if and only if $2A \geq \alpha C(1-d)\left[(\beta+\theta+r)e^{-rM}+i\right]T_{cd}^2$ (26)

The optimal present value of all future cash flows can be obtained as

$$APV_4\left(T_4^*\right) \approx \frac{1}{\left(1-e^{-rT_4^*}\right)}\left\{A+\frac{1}{(\beta+\theta+r)}\left[\sqrt{2AC\alpha(1-d)\left[(\beta+\theta+r)e^{-rM}+i\right]}+\theta\right]\right\} - \frac{iC\alpha(1-d)}{r\left(\beta+\theta+r\right)} \tag{27}$$

5. Comparisons among different policies:

By comparing the optimal replenishment cycles, ordered quantities and present values of all future cash flows among these four cases, we obtain the following result.

Theorem 2

(a) If $\left[\left(\beta+\theta+r\right)e^{-rN}+i\right]>\left(1-d\right)\left[\left(\beta+\theta+r\right)e^{-rM}+i\right]$, then $T_1^*=T_3^*<T_2^*<T_4^*$ and $Q_1^*=Q_3^*<Q_2^*<Q_4^*$

(b) If $\left[\left(\beta+\theta+r\right)e^{-rN}+i\right]=\left(1-d\right)\left[\left(\beta+\theta+r\right)e^{-rM}+i\right]$, then $T_1^*=T_3^*<T_2^*=T_4^*$ and $Q_1^*=Q_3^*<Q_2^*=Q_4^*$

(c) If $\left[\left(\beta+\theta+r\right)e^{-rN}+i\right]<\left(1-d\right)\left[\left(\beta+\theta+r\right)e^{-rM}+i\right]$, then $T_1^*=T_3^*<T_4^*<T_2^*$ and $Q_1^*=Q_3^*<Q_4^*<Q_2^*$

Proof.

If $\left[\left(\beta+\theta+r\right)e^{-rN}+i\right]>\left(1-d\right)\left[\left(\beta+\theta+r\right)e^{-rM}+i\right]$, then from N > M, we obtain $\left[\left(\beta+\theta+r\right)e^{-rM}+i\right]>\left[\left(\beta+\theta+r\right)e^{-rN}+i\right]$. Using Eq. (16), (19), (22) and (25), we can easily obtain $T_1^*=T_3^*<T_2^*<T_4^*$. Then, by Eq. (3), $Q_j^*=\frac{\alpha}{(\beta+\theta)}\left[e^{(\beta+\theta)T_j^*}-1\right]$, j =1, 2, 3, 4 is an increasing function of T_j^*. Thus, $Q_1^*=Q_3^*<Q_2^*<Q_4^*$. This completes the proof of (a). By the similar procedure as described in (a), the proofs of (b) and (c) can be easily completed.

6. Comparisons among different policies

There are three situations: (A) $Q_{pd}>Q_{cd}$; (B) $Q_{pd}<Q_{cd}$; (C) $Q_{pd}=Q_{cd}$ for opting the optimal alternative based on the values of Q_{pd} and Q_{cd}. Now, we will talk about how to use the features of the optimal solution in order to find the optimal alternative easily. For notational convenience,

Let $\Delta_1=C\alpha\left[(\beta+\theta+r)e^{-rM}+i\right]T_{pd}^2$, $\Delta_2=C\alpha\left[(\beta+\theta+r)e^{-rN}+i\right]T_{pd}^2$, $\Delta_3=C\alpha\left[(\beta+\theta+r)e^{-rM}+i\right]T_{cd}^2$ and $\Delta_4=C\alpha(1-d)\left[(\beta+\theta+r)e^{-rM}+i\right]T_{cd}^2$, Hence we have $\Delta_1>\Delta_2$ and $\Delta_3>\Delta_4$.

6.1. Situation A: $Q_{pd}>Q_{cd}$

This takes place when the minimum order quantity with a longer permissible delay in payment N is larger than that with cash discount rate d. Furthermore, we get $\Delta_1>\Delta_3$. Using Eq. (17), (20), (23) and (26) the replenishment cycle T* that minimizes the present value of all future cash flows can be obtained. Thus, we have the following results, for which the proof is trivial and hence we omit it here.

Theorem 3. When $Q_{pd}>Q_{cd}$

Situation A	Condition	APV (T*)	T*
$\Delta_2>\Delta_3$	$2A<\Delta_4$	min{APV₁(T_1^*),APV₃(T_3^*),APV₄(T_{cd})}	T_1^*, T_3^* or T_{cd}
	$\Delta_4\le 2A<\Delta_3$	min{APV₁(T_1^*),APV₃(T_3^*),APV₄(T_4^*)}	T_1^*, T_3^* or T_4^*
	$\Delta_3\le 2A<\Delta_2$	min{APV₁(T_1^*),APV₂(T_{pd}),APV₄(T_4^*)}	T_1^*, T_{pd} or T_4^*
	$2A\ge\Delta_2$	min{APV₂(T_2^*),APV₄(T_4^*)}	T_2^* or T_4^*
$\Delta_3>\Delta_2>\Delta_4$	$2A<\Delta_4$	min{APV₁(T_1^*),APV₃(T_3^*),APV₄(T_{cd})}	T_1^*, T_3^* or T_{cd}
	$\Delta_4\le 2A<\Delta_2$	min{APV₁(T_1^*),APV₃(T_3^*),APV₂(T_{pd}),APV₄(T_4^*)}	T_1^*, T_3^*, T_{pd} or T_4^*

	$2A \geq \Delta_2$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_4^*)\}$	T_2^* or T_4^*
$\Delta_4 > \Delta_2$	$2A < \Delta_2$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_3(T_3^*), \mathrm{APV}_2(T_{pd})\}$	T_1^*, T_3^*, or T_{pd}
	$2A \geq \Delta_2$	$\mathrm{APV}_2(T_2^*)$	T_2^*

6.2. Situation B: $Q_{pd} < Q_{cd}$

This occurs when the minimum order quantity with a longer permissible delay in payment Nis smaller than that with cash discount rate d, hence, $\Delta_1 < \Delta_3$. By Eq. (17), (20), (23) and (26) the replenishment cycle T* that minimizes the present value of all future cash flows can be obtained and is given as follows. The proof is omitted.

Theorem 4. When $Q_{pd} < Q_{cd}$

Situation B	Condition	APV (T*)	T*
$\Delta_1 > \Delta_4 > \Delta_2$	$2A < \Delta_2$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_3(T_3^*), \mathrm{APV}_2(T_{pd})\}$	T_1^*, T_3^* or T_{pd}
	$\Delta_2 \leq 2A < \Delta_4$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_{cd})\}$	T_2^* or T_{cd}
	$2A \geq \Delta_4$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_4^*)\}$	T_2^* or T_4^*
$\Delta_4 > \Delta_1 > \Delta_2$	$2A < \Delta_2$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_3(T_3^*), \mathrm{APV}_2(T_{pd})\}$	T_1^*, T_3^* or T_{pd}
	$\Delta_2 \leq 2A < \Delta_1$	$\mathrm{APV}_2(T_2^*)$	T_2^*
	$\Delta_1 \leq 2A < \Delta_4$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_{cd})\}$	T_2^* or T_{cd}
	$2A \geq \Delta_4$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_4^*)\}$	T_2^* or T_4^*
$\Delta_4 \leq \Delta_2$	$2A < \Delta_4$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_3(T_3^*), \mathrm{APV}_4(T_{cd})\}$	T_1^*, T_3^*, or T_{cd}
	$\Delta_4 \leq 2A < \Delta_2$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_3(T_3^*), \mathrm{APV}_2(T_{pd}), \mathrm{APV}_4(T_4^*)\}$	T_1^*, T_3^*, T_{pd} or T_4^*
	$2A \geq \Delta_2$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_4^*)\}$	T_2^* or T_4^*

6.2. Situation C: $Q_{pd} = Q_{cd}$

This means the minimum order quantity with a longer permissible delay in payment N is the same as that with cash discount rate d, thus, $T_{pd} = T_{cd}$ and then $\Delta_1 = \Delta_3$. Using Eq. (17), (20),(23) and (26) the replenishment cycle T* that minimizes the present value of all future cash flows can be obtained and is given as follows. The proof is omitted.

Theorem 5. When $Q_{pd} = Q_{cd}$

Situation C	Condition	APV (T*)	T*
$\Delta_2 > \Delta_4$	$2A < \Delta_4$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_4(T_{cd})\}$	T_1^* or T_{cd}
	$\Delta_4 \leq 2A < \Delta_2$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_2(T_{pd}), \mathrm{APV}_4(T_4^*)\}$	T_1^*, T_{pd} or T_4^*
	$2A \geq \Delta_2$	$\min\{\mathrm{APV}_2(T_2^*), \mathrm{APV}_4(T_4^*)\}$	T_2^* or T_4^*
$\Delta_4 > \Delta_2$	$2A < \Delta_2$	$\min\{\mathrm{APV}_1(T_1^*), \mathrm{APV}_2(T_{pd})\}$	T_1^* or T_{pd}

$2A \geq \Delta_2$	APV$_2$(T_2^*)	T_2^*

7. Numerical Examples:

Example 1. Consider, the parametric values α = 1000 units, β = 0.05, C = 20, θ = 0.15, i = 0.2,r = 0.05, d = 0.05, M = 15/365 year = 15 days, N = 30/365 year = 30 days. If $Q_{pd} = 100$ units and $Q_{cd} = 80$ units, we know that $Q_{pd} > Q_{cd}$, $T_{pd} = 0.099013$, $T_{cd} = 0.0793667$, $\Delta_1 = 88.13179$, $\Delta_2 = 88.03138$, $\Delta_3 = 56.62707$, $\Delta_4 = 53.79572$. By using Theorem 3, we obtain the computational results for different values of A as shown in Table 2.

Example 2. By replacing N = 45/365 year = 45 days and $Q_{cd} = 150$ units, in Example 1, we obtain that $Q_{pd} < Q_{cd}$, $T_{pd} = 0.099013$, $T_{cd} = 0.147794$, $\Delta_1 = 88.13179$, $\Delta_2 = 87.931174$, $\Delta_3 = 196.36344$, $\Delta_4 = 186.54527$. By using Theorem 4, we obtain the computational results for different values of A as shown in Table 3.

Example 3. By substituting $Q_{cd} = 100$ units, in Example 1, we obtain that $Q_{pd} = Q_{cd}$, $T_{pd} = T_{cd} = 0.099013$, $\Delta_1 = 88.13179$, $\Delta_2 = 88.03138$, $\Delta_3 = 88.13179$, $\Delta_4 = 83.72520$. By using Theorem 5, we obtain the computational results for different values of A as shown in Table 4.

Example 4. Replacing A = 35 and $Q_{cd} = 120$ units, in Example 1, we obtain $Q_{pd} < Q_{cd}$, $T_{pd} = 0.02849$, $T_{cd} = 0.03419$. The computational results for different values of d and N are shown in Table 5

Table – 2 Optimal solution for different ordering costs A ($Q_{pd} > Q_{cd}$)

A	T*	Q*	APV*
10	Ty=0.079367	80	9633731.974
35	T4=0.090534	91.35904	17730565.74
45	T4=0.102656	103.71747	20068806.63
60	T4=0.118537	119.95367	23138841.42
70	T4=0.128035	129.688385	24978702.77

Table – 3 Optimal solution for different ordering costs A ($Q_{pd} < Q_{cd}$)

A	T*	Q*	APV*
10	T1=T3=0.047167	47.39053	9872356.303
35	T1=T3=0.088242	89.0253	18177159.04
45	Ty=0.147794	150	20068806.63
60	Ty=0.147794	150	23138841.42
70	Ty=0.147794	150	24978702.77

Table – 4 Optimal solution for different ordering costs A ($Q_{pd} = Q_{cd}$)

A	T*	Q*	APV*
10	Ty=0.09901	100	9633731.974
35	Ty=0.09901	100	17730565.74
45	T4=0.102656	103.7175	20068806.63
60	T4=0.118537	119.95366	23138841.42
70	T4=0.128035	129.68839	24978702.77

Table – 5 Optimal solution for different cash discount rate d and N

N	d	T*	Q*	APV*
30	0.01	T4=0.047405	166.706	116226082.2

	0.05	T4=0.0483927	170.197	111587462.7
	0.1	T4=0.0497188	174.884	105789188.4
	0.15	T4=0.051160	179.98	99990914.14
60	0.01	T4=0.047405	166.706	116811000.9
	0.05	T4=0.0483927	170.197	114462989.8
	0.1	T4=0.0497188	174.884	111461299.1
	0.15	T4=0.051160	179.98	108374136.5
90	0.01	T4=0.047405	166.706	116226082.2
	0.05	T4=0.0483927	170.197	111587462.7
	0.1	T4=0.0497188	174.884	105789188.4
	0.15	T4=0.051160	179.98	99990914.14

Table -6 Sensitivity analysis for α ($Q_{pd} > Q_{cd}$)

α	A	T*	Q*	APV*
800	10	Ty=0.099013	80	6931099.17
	35	T4=0.101220	81.8	12732651.78
	45	T4=0.114771	92.88	14409220.68
	60	T4=0.132529	107.441	16610620.45
	70	T4=0.143148	116.173	17929992.21
1000	10	Ty=0.079367	80	9633731.974
	35	T4=0.090534	91.35904	17730565.74
	45	T4=0.102656	103.71747	20068806.63
	60	T4=0.118537	119.95367	23138841.42
	70	T4=0.128035	129.688385	24978702.77
1200	10	Ty=0.066220	79.999	12612793.89
	35	T4=0.082646	99.9996	23245184.45
	45	T4=0.093710	113.5149	26315218.71
	60	T4=0.108209	131.2665	30344326.43
	70	T4=0.141908	141.9075	32758634.08

Table -7 Sensitivity analysis for α ($Q_{pd} < Q_{cd}$)

α	A	T*	Q*	APV*
800	10	T1=0.052735	42.411	7102303.546
	35	T1=0.098658	79.7098	13052390.65
	45	T1=0.111867	150	14771892.65
	60	Ty=0.184070	150	16610620.45
	70	Ty=0.184070	150	17929992.21
1000	10	T1=T3=0.047167	47.39053	9872356.303
	35	T1=T3=0.088242	89.0253	18177159.04
	45	Ty=0.147794	150	20068806.63
	60	Ty=0.147794	150	23138841.42
	70	Ty=0.147794	150	24978702.77
1200	10	T1=0.043058	51.8924	12927740.12
	35	T1=0.080551	97.4472	23832568.23
	45	Ty=0.123463	150	26315218.71
	60	Ty=0.123463	150	30344326.43
	70	Ty=0.123463	150	32758634.08

Table -8 Sensitivity analysis for α ($Q_{pd} = Q_{cd}$)

α	A	T*	Q*	APV*
800	10	Ty=0.123463	100	6931099.17
	35	Ty=0.123463	100	12732651.78
	45	Ty=0.123463	125	14409220.68

	60	T4=0.132529	107.44	16610620.45
	70	T4=0.1431475	116.173	17929992.21
1000	10	Ty=0.09901	100	9633731.974
	35	Ty=0.09901	100	17730565.74
	45	T4=0.102656	103.7175	20068806.63
	60	T4=0.118537	119.95366	23138841.42
	70	T4=0.128035	129.68839	24978702.77
1200	10	Ty=0.082647	100	12612793.89
	35	Ty=0.082647	100	23245184.45
	45	T4=0.09371	113.51	26315218.71
	60	T4=0.10821	131.267	30344326.43
	70	T4=0.116879	141.907	32758634.08

Table -9 Sensitivity analysis for β ($Q_{pd} > Q_{cd}$)

β	A	T*	Q*	APV*
0.03	10	Ty=0.07943	80	10891025.99
	35	T4=0.09261	93.39	20087366.33
	45	T4=0.105014	106.013	22742307.82
	60	T4=0.121260	122.593	26227574.57
	70	T4=0.130976	132.532	28315770.91
0.05	10	Ty=0.079367	80	9633731.974
	35	T4=0.090534	91.35904	17730565.74
	45	T4=0.102656	103.71747	20068806.63
	60	T4=0.118537	119.95367	23138841.42
	70	T4=0.128035	129.688385	24978702.77
0.07	10	Ty=0.0793042	80	8616638.74
	35	T4=0.088589	89.458	15825080.79
	45	T4=0.114504	101.5685	17907406.98
	60	T4=0.11599	117.483	20641694.44
	70	T4=0.12528	127.026	22280240.79

Table -10 Sensitivity analysis for β ($Q_{pd} < Q_{cd}$)

β	A	T*	Q*	APV*
0.03	10	T1=0.04825	48.461	11163806.8
	35	T1=0.090269	91.006	20595382.49
	45	Ty=0.1480107	150	22742307.82
	60	Ty=0.1480107	150	26227574.57
	70	Ty=0.1480107	150	28315770.91
0.05	10	T1=T3=0.047167	47.39053	9872356.303
	35	T1=T3=0.088242	89.0253	18177159.04
	45	Ty=0.147794	150	20068806.63
	60	Ty=0.147794	150	23138841.42
	70	Ty=0.147794	150	24978702.77
0.07	10	T1=0.046154	46.389	8828989.953
	35	T1=0.086346	87.171	16222441.11
	45	T1=0.097907	98.969	18358349.62
	60	Ty=0.147578	150	20124863.3
	70	Ty=0.147578	150	21721189.95

Table -11 Sensitivity analysis for β ($Q_{pd} = Q_{cd}$)

β	A	T*	Q*	APV*
0.03	10	Ty=0.990112	100	10891025.99
	35	Ty=0.990112	100	20087366.33

	45	T4=0.1050144	106.013	22742307.82
	60	T4=0.121260	122.593	26227574.57
	70	T4=0.1309759	132.532	28315770.91
0.05	10	Ty=0.09901	100	9633731.974
	35	Ty=0.09901	100	17730565.74
	45	T4=0.102656	103.7175	20068806.63
	60	T4=0.118537	119.95366	23138841.42
	70	T4=0.128035	129.68839	24978702.77
0.07	10	Ty=0.098916	100	8616638.74
	35	Ty=0.098916	100	15825080.79
	45	T4=0.100450	101.5686	17907406.98
	60	T4=0.115990	117.483	20641694.44
	70	T4=0.12528	127.026	22280240.79

Table 12 Sensitivity analysis for θ ($Q_{pd} > Q_{cd}$)

θ	A	T*	Q*	APV*
0.05	10	Ty=0.0796817	80	21299976.95
	35	T4=0.10264299	103.172	39580352.31
	45	T4=0.1163862	117.066	44852360.46
	60	T4=0.1343912	135.298	51768053.27
	70	T4=0.145159	146.218	55909419.79
0.15	10	Ty=0.079367	80	9633731.974
	35	T4=0.090534	91.35904	17730565.74
	45	T4=0.102656	103.71747	20068806.63
	60	T4=0.118537	119.95367	23138841.42
	70	T4=0.128035	129.688385	24978702.77
0.25	10	Ty=0.07906	80	5978485.739
	35	T4=0.0818983	82.913	10879946.09
	45	T4=0.092864	94.17	12297359.03
	60	T4=0.10723	108.973	14159578.07
	70	T4=0.115822	117.857	15276001.93

Table 13 Sensitivity analysis for θ ($Q_{pd} < Q_{cd}$)

θ	A	T*	Q*	APV*
0.05	10	T1=0.053476	53.619	21842509.49
	35	Ty=0.148886	150	39580352.31
	45	Ty=0.148886	150	44852360.46
	60	Ty=0.148886	150	51768053.27
	70	Ty=0.148886	150	55909419.79
0.15	10	T1=T3=0.047167	47.39053	9872356.303

	35	T1=T3=0.088242	89.0253	18177159.04
	45	Ty=0.147794	150	20068806.63
	60	Ty=0.147794	150	23138841.42
	70	Ty=0.147794	150	24978702.77
0.25	10	T1=0.042668	42.942	6122487.872
	35	T1=0.079800	80.788	11149570.01
	45	T1=0.09051	91.753	12603073.22
	60	Ty=0.14672	150	14159578.07
	70	Ty=0.14672	150	15276001.93

Table 14 Sensitivity analysis for θ ($Q_{pd} = Q_{cd}$)

θ	A	T*	Q*	APV*
0.05	10	Ty=0.099503	100	21299976.95
	35	T4=0.102643	103.1716	39580352.31
	45	T4=0.11639	117.06614	44852360.46
	60	T4=0.134391	135.298	51768053.27
	70	T4=0.145159	146.2178	55909419.79
0.15	10	Ty=0.09901	100	9633731.974
	35	Ty=0.09901	100	17730565.74
	45	T4=0.102656	103.7175	20068806.63
	60	T4=0.118537	119.95366	23138841.42
	70	T4=0.128035	129.68839	24978702.77
0.25	10	Ty=0.098529	100	59784857.39
	35	Ty=0.098529	100	10879946.09
	45	Ty=0.098529	100	12297359.03
	60	T4=0.10723	108.97	14159578.07
	70	T4=0.115822	117.857	15276001.93

7. Conclusions:

Appendix A

To prove Theorem 1a, we set the left-hand side of (13) as

$$D_j\left(T_j\right) = \left(\beta+\theta\right)\delta_{2j}\, e^{(\beta+\theta+r)T_j} - \left(\beta+\theta+r\right)\delta_{2j}\, e^{(\beta+\theta)T_j}\;; \quad \text{j = 1, 2, 3, 4.}$$

Taking the first derivative of $D_j\left(T_j\right)$ with respect to T_j, we get

$$\frac{dD_j\left(T_j\right)}{dT_j} = \left(\beta+\theta\right)\left(\beta+\theta+r\right)\delta_{2j}\, e^{(\beta+\theta)T_j}\left(e^{-rT_j}-1\right) > 0$$

Hence, $D_j\left(T_j\right)$ is a strictly increasing function in T_j, for j = 1, 2, 3, 4.

Furthermore, $D_j(0) = -r\delta_{2j} < 0$, and $\lim\limits_{T_j\to\infty} D_j\left(T_j\right) = \infty$. Therefore, there exists a unique solution $T_j^* \in (0,\infty)$ such that $D_j\left(T_j^*\right) = \delta_{1j}\, r + \frac{iC(1-d)\alpha}{(\beta+\theta+r)}$, that is, the solution $T_j^* \in (0,\infty)$ which satisfies Eq. (13) not only exists but also is unique. This completes the proof.

To prove Theorem 1b, we simply check the second-order condition at point T_j^*. Taking the second derivative of $APV_j\left(T_j\right)$ with respect to T_j^* and then substituting $T_j = T_j^*$ into this equation, we obtain

$$\left.\frac{d^2 APV_j(T_j)}{d^2 T_j}\right|_{T_j=T_j^*} = \frac{-2re^{-rT_j^*}}{\left(1-e^{-rT_j^*}\right)^3}\left\{-\left(\delta_{1j}+\delta_{2j}\,e^{(\beta+\theta)T_j^*}\right)re^{-rT_j^*}+\left(1-e^{-rT_j^*}\right)\delta_{2j}(\beta+\theta)\,e^{(\beta+\theta)T_j^*}-\frac{iC(1-d_j)\alpha}{(\beta+\theta+r)}e^{-rT_j^*}\right\}$$

$$+\frac{1}{\left(1-e^{-rT_j^*}\right)^2}\left\{\left(\delta_{1j}+\delta_{2j}\,e^{(\beta+\theta)T_j^*}\right)r^2e^{-rT_j^*}+\left(1-e^{-rT_j^*}\right)\delta_{2j}(\beta+\theta)^2\,e^{(\beta+\theta)T_j^*}+\frac{iC(1-d_j)\alpha}{(\beta+\theta+r)}e^{-rT_j^*}\right\}$$

$$=\frac{1}{\left(1-e^{-rT_j^*}\right)^2}\left\{(\beta+\theta+r)\left(1-e^{-rT_j^*}\right)(\beta+\theta)\delta_{2j}\,e^{(\beta+\theta)T_j^*}\right\}$$

(by using (13)) > 0

This completes the proof

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