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Extension Of the Conjecture Of Gratzer

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Abstract

In this paper we extend two results of Gratzer on Distributive Lattice.

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1. Introduction:

Gratzer conjectured that If a lattice has a representation of Type 2 then it is modular, and the linear subspaces of a projective space form a Modular geometric Lattice. . In this paper we establish this conjecture for distributive lattices.

We first give some terms which are useful in this paper.:

(1.1) **Representation of Type 2: [1, P.197]:** A representation $\alpha : L \rightarrow$ Part (A) is called of *type 2* iff for all $a, b \in L$ and $x, y \in A$
$x \equiv y\, (a \vee b)\, \alpha)$ iff there exist $z_1, z_2 \in A$ such that
$x \equiv z_1\, (a\, \alpha)$, $z_1 \equiv z_2\, (b\, \alpha)$, and $z_1 \equiv y\, (a\, \alpha)$,

(1.2) **Projective space [1,P.202]:** If A be a set and L be a collection of subset of A. (A, L) is called Projective space iff the following properties hold:

(i) Every $l \in L$ has at least two elements.

(ii) For any two distinct p and q $\in A$ there is exactly one $l \in L$ satisfying p and $q \in L$.

(iii) For p, q, r, x, y $\in A$ and $l_1, l_2 \in L$ satisfying p, q, x $\in l_1$ and q, r, y $\in l_2$. there exist $z \in A$ and $l_3, l_4 \in L$ satisfying p, r, z $\in l_3$ and x, y, z $\in l_4$.

(1.3) **Linear subspace** [1,P203]: A set $X \subseteq A$ is called linear subspace iff p and q $\in X$ imply that $p + q \subseteq X$. If X and Y are linear subspaces then define
$X + Y = \cup\, (x + y \mid x \in X \text{ and } y \in Y)$.

2. Main Theorems:

Theorem (2.1) *A Lattice L has a representation of type 2, then it is Distributive.*

Proof: Let L have a representation $\alpha : L \rightarrow part(A)$ of type 2, and $a, b, c \in L$, $a \geq c$.

Since, in any lattice ($(a \wedge b) \vee c \leq (a \vee c) \wedge (b \vee c)$.

we have to prove

$(a \wedge b) \vee c \geq (a \vee c) \wedge (b \vee c)$. so let x,y $\in A$. and

let x $\equiv y\,((a \wedge (b \vee c))\alpha)$, that is

$x \equiv y(a\alpha)$ and $x \equiv y((b \vee c)\alpha$.

As α is a type 2 representation there exist z_1 and z_2 such that

x $\equiv$ z_1 (cα) . $z_1 \equiv z_2$ (bα) And $z_2 \equiv$ y(cα)

Since $c \leq a$,we obtain that that $z_1 \equiv$ x($a\alpha$), x $\equiv$ y ($a\alpha$) and y $\equiv$ z_2 (aα);

thus $z_1 \equiv z_2$ (aα)

Also $z_1 \equiv z_2$ (bα),hence $z_1 \equiv z_2\,((a \wedge b)\alpha)$ and $x \equiv y(a\alpha)$

Hence x $\equiv y\,(((a \wedge b) \vee (a \wedge c))\alpha)$, implying $(a \wedge b) \vee c \geq (a \vee c) \wedge (b \vee c)$.

Theorem (2.2): *The linear subspaces of a projective space form a Distributive geometric Lattice.*

Proof: Since the intersection of any number of linear subspaces is a linear subspace again, we have a closure space *(A* **,** -). For $X \subseteq$ ***A*** **,** the closure *x* can be described as follows: set $Xo = X, Xi = X + X, \ldots, Xn = Xn\text{-}l + Xn\text{-}l, \ldots$;
then $\overline{X} = \bigcup (X_i$ */i=0,1,2,.....)*

It follows immediately, that ***(A* , -)** is an algebraic closure space and so the linear subspaces form an algebraic lattice and for the linear subspaces *X* and *Y* **,**

$$X \vee Y = \overline{X \cup Y}.$$

If *X , Y ,* and *Z* are linear subspaces and Z $\subseteq X$ then

$$(X \wedge Y) \vee (X \wedge Z) \subseteq X \wedge (Y \vee Z).$$

Now let p $\in X \wedge (Y \vee Z), i.e. p \in X$ and p $\in Y \vee Z$ Since $p \in Y \vee Z = Y + Z$, there exist $p_y \in Y$ and $p_z \in Z$ such that p $\in$ $p_y + p_z$.. from Z $\subseteq X$, it is clear that p and $p_z \in$ X, if p= p_z ,then p $\in z$ then so $p \in (x \wedge y) \vee (x \vee z)$.*if* p $\neq$ p_z,then $p_y \in p +$ $p_z \subseteq X$.
thus $p_y \in X \wedge Y$ and $p_z \in (X \wedge Z)$. thus given lattice is distributive.

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