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Boundary Value Problems involving Flow of Multi-layered Fluid over Undulating Bottom in a Channel

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Abstract

A class of mixed boundary value problems (BVPs) arising in the study of fluid flow involving three layers of fluids in a channel associated with small undulation on the bottom, is examined for the solution using two-dimensional linearized theory. Here it is assumed that the uppermost fluid is free to the atmosphere. Using the perturbation analysis in conjunction with the Fourier transform technique, the velocity potentials and the elevations at each fluid layer are obtained in terms of first-order of a small undulation parameter h introduced in the description of the bottom topography. The special case of a patch of sinusoidal ripples on the bed is handled in detail and the numerical results of these physical quantities are demonstrated graphically.

Keywords: Linear theory, Irrotational flow, Perturbation technique, Fourier

transform

1 Introduction

The problem of free surface fluid flow over submerged obstacles have created varieties of challenges to model the situations in engineering, atmospheric and oceanographic sciences. Such problems were considered for their complete solution by Dias and Vanden-Broeck [1, 2], Forbes [3], Forbes and Schwartz [4], Martha and Bora [5], Martha et.al. [6], Shen et.al. [7], Vanden-Broeck [8], and many others. The problem involving two layers of fluids where the fluid in each layer is inviscid and incompressible, were handled by Belward and Forbes [9], Dias and Vanden-Broeck [10, 11], Chakrabarti and Martha [12] assuming the uppermost fluid layer is a rigid lid.

In this paper, we consider the flow problem involving three layers of fluids of different constant densities in a channel associated with small undulation on the bottom where the uppermost fluid layer is free to atmosphere. The solution of this problem is obtained by employing perturbation analysis in conjunction with the Fourier transform technique. Analytical expressions of the elevation of the interfaces are derived within the framework of two-dimensional linearized theory. One special case for the bottom profile is considered to workout the expressions in detail. The numerical results of these physical quantities are presented graphically.

2 Description of the problem

We consider a system involving three layers of fluids flowing over bottom topography having a small undulation in a channel. We assume that the fluid in each layer is inviscid, incompressible and have constant but different densities. The flow in each layer is two dimensional, irrotational with the far upstream velocity uniform. The profile of the bottom topography is given by $y = B(x)$ where the x-axis is chosen to be along the bottom of the channel and y-axis is chosen in the vertically upward direction. The upper fluid layer is free to atmosphere. We let subscript 1 refer to quantities related in the upper layer, subscript 2 refer to quantities in the middle layer and subscript 3 refer to quantities in the lower layer. We then denote the densities by ρ_j, velocities by q_j, pressures by p_j, upstream depth by H_j and upstream horizontal velocity by c_j in each layer at any point (x_j, y_j), $j = 1, 2, 3$. The interface between layers 1 and 2 is denoted by $y = Q(x)$ and the interface between layers 2 and 3 is denoted by $y = S(x)$, the free surface is represented by $y = P(x)$. Here these two interfaces $y = Q(x)$ and $y = S(x)$ are unknown at the outset, to be determined.

Let $\phi_j, (j = 1, 2, 3)$ be the velocity potential in layer j. So $\vec{q_j} = (u_j, v_j) = (\phi_{j,x}, \phi_{j,y})$ where $\phi_{j,x}$, $\phi_{j,y}$ denote the partial derivatives of ϕ_j with respect to x and y respectively. In the following sections $\phi_{j,xx}$, $\phi_{j,yy}$ denote the second order partial derivatives of ϕ_j with respect to x and y.

The above variables are non-dimensionalised using H_3 as the length scale and c_3 as the velocity scale. So the lower layer has an upstream uniform speed of 1 and upstream uniform height of 1. The dimensionless quantities which represent the properties of the flow,

$$\lambda_1 = \frac{H_1}{H_3},\ \lambda_2 = \frac{H_2}{H_3} \text{ (ratio of upstream depths of fluids);}$$
$$D_1 = \frac{\rho_1}{\rho_2}, \quad D_2 = \frac{\rho_2}{\rho_3}, \text{(the ratio of densities);}$$
$$\gamma_1 = \frac{c_1}{c_3}, \quad \gamma_2 = \frac{c_2}{c_3} \quad \text{(the ratio of upstream speeds);}$$
$$F_3 = \frac{c_3}{\sqrt{gH_3}} \quad \text{(Froude number in the lower layer)}$$

and the two dimensionless parameters which describe the properties of the obstacle, h the obstacle height and L the obstacle half length. The following work proceeds purely with non-dimensionalised variables.

3 Mathematical Formulation

We assume the propagation of stationary waves with respect to the bottom profile, so that the partial derivatives with respect to time can be taken equal to zero. Then within each layer, the velocity potential $\phi_j,\ (j = 1, 2, 3)$ satisfy the Laplace's equation:

$$\nabla^2 \phi_j = 0. \tag{1}$$

The conditions on the free surface are given by

$$\left.\begin{array}{l} \phi_{1,n} = 0, \\ \frac{1}{2}F_3^2(q_1^2 - \gamma_1^2) + P(x) = 1 + \lambda_1 + \lambda_2, \end{array}\right\} \text{ on } \ y = P(x), \tag{2}$$

since there is no fluid exchange at the interfaces, the conditions at the interfaces are:

$$\phi_{j,n} = 0, \qquad \text{on } \ y = Q(x), \quad j = 1, 2, \tag{3}$$
$$\phi_{j,n} = 0, \qquad \text{on } \ y = S(x), \quad j = 2, 3, \tag{4}$$

and the condition at the bottom is:

$$\phi_{3,n} = 0, \qquad \text{on } \ y = B(x). \tag{5}$$

Here, in the above equations, $\partial/\partial n$ means the normal derivative at a point (x, y) at the respective surfaces.
At the interfaces, continuity of pressure, coupled with Bernoulli equation gives the matching condition:

$$\frac{F_3^2}{2}(q_2^2 - D_1 q_1^2) + (1 - D_1)Q(x) =$$

$$\frac{F_3^2}{2}(\gamma_2^2 - D_1\gamma_1^2) + (1+\lambda_2)(1-D_1) \quad \text{on} \quad y = Q(x), \tag{6}$$

$$\frac{1}{2}F_3^2(q_3^2 - D_2 q_2^2) + (1-D_2)S(x) = \frac{1}{2}F_3^2(1 - D_2\gamma_2^2) + (1-D_2) \quad \text{on } y = S(x). \tag{7}$$

The upstream conditions are

$$\overrightarrow{q_1} \to \gamma_1 \overrightarrow{i}, \ \overrightarrow{q_2} \to \gamma_2 \overrightarrow{i}, \ \overrightarrow{q_3} \to \overrightarrow{i}, \ P(x) \to 1+\lambda_1+\lambda_2,$$
$$Q(x) \to 1+\lambda_2, \ S(x) \to 1 \text{ as } x \to -\infty.$$

In the next section, the solutions of the BVPs involving equations (1)-(7) are determined by the help of perturbation analysis with Fourier transform technique.

4 Method of Solution

Here, we assume that the bottom profile is given by $y = B(x) = hf(x)$ (say), where h is the height of the bottom profile, a dimensionless small quantity. We then express the velocity potentials, the upper surface and the interfaces as the regular perturbations:

$$\phi_1(x,y) = \gamma_1 x + h\phi_{11}(x,y) + O(h^2), \tag{8}$$
$$\phi_2(x,y) = \gamma_2 x + h\phi_{21}(x,y) + O(h^2), \tag{9}$$
$$\phi_3(x,y) = x + h\phi_{31}(x,y) + O(h^2), \tag{10}$$
$$P(x) = 1+\lambda_2+\lambda_1 + hP_1(x) + O(h^2), \tag{11}$$
$$Q(x) = 1+\lambda_2 + hQ_1(x) + O(h^2), \tag{12}$$
$$S(x) = 1 + hS_1(x) + O(h^2). \tag{13}$$

Substituting these equations (8)-(13) in the equations (1)-(7), we get (up to order h):

$$\left.\begin{array}{ll}
\nabla^2(\phi_{11},\phi_{21},\phi_{31}) = 0, & \text{(within each layer)} \\
\phi_{11,y} = \gamma_1 P_1'(x), & \text{on } y = 1+\lambda_1+\lambda_2 \\
\phi_{11,y} = \gamma_1 Q_1'(x), & \text{on } y = 1+\lambda_2 \\
\phi_{21,y} = \gamma_2 Q_1'(x), & \text{on } y = 1+\lambda_2 \\
\phi_{21,y} = \gamma_2 S_1'(x), & \text{on } y = 1 \\
\phi_{31,y} = S_1'(x), & \text{on } y = 1 \\
\phi_{31,y} = f'(x), & \text{on } y = 0 \\
F_3^2\gamma_1\phi_{11,x} + P_1(x) = 0 & \text{on } y = 1+\lambda_1+\lambda_2 \\
F_3^2\left[\gamma_2\phi_{21,x} - D_1\gamma_1\phi_{11,x}\right] & \\
\qquad +(1-D_1)Q_1(x) = 0 \ \text{ on } y = 1+\lambda_2 & \\
F_3^2\left[\phi_{31,x} - D_2\gamma_2\phi_{21,x}\right] & \\
\qquad +(1-D_2)S_1(x) = 0 \ \text{ on } y = 1 &
\end{array}\right\} \tag{14}$$

To solve the boundary value problems involving relation (14), we now assume that the first order potentials $\phi_{j1}(x,y), j = 1,2,3$ and the bottom profile $f(x)$

are such that the Fourier transforms of $\phi_{j1}(x,y)$ and $f(x)$ exist, and are defined as follows:

$$\widehat{\phi}_{j1}(k,y) = \frac{2}{\pi}\int_0^\infty \phi_{j1}(x,y)\sin(kx)dx, \tag{15}$$

with the inverse transform

$$\phi_{j1}(x,y) = \int_0^\infty \widehat{\phi}_{j1}(k,y)\sin(kx)dk, \tag{16}$$

and

$$f(x) = \int_0^\infty M(k)\cos(kx)dk, \tag{17}$$

with the inverse transform

$$M(k) = \frac{2}{\pi}\int_0^\infty f(x)\cos(kx)dx. \tag{18}$$

Now let's define $Q_1(x)$ and $S_1(x)$ as

$$Q_1(x) = \int_0^\infty b(k)\cos(kx)dk, \tag{19}$$

$$S_1(x) = \int_0^\infty a(k)\cos(kx)dk. \tag{20}$$

The interfaces $Q(x)$ and $S(x)$ will be determined once the unknowns $b(k)$ and $a(k)$ are determined.
Applying these transforms to the BVPs involving relation (14) and solving them, we obtain

$$b(k) = \frac{F_3^4\gamma_2^2 kM(k)E_1(k)\sinh(k\lambda_2)}{E_2(k)}, \tag{21}$$

$$a(k) = \frac{F_3^2 kM(k)E_3(k)\sinh(k\lambda_2)}{E_2(k)}, \tag{22}$$

where

$$\begin{aligned} E_1(k) &= [F_3^2\gamma_1^2 k\cosh(k\lambda_1) - \sinh(k\lambda_1)]/k, \\ E_2(k) &= E_3(k)\big[F_3^2 k\cosh k\sinh k\lambda_2 + \big\{\gamma_2^2F_3^2kD_2\cosh k\lambda_2 \\ &\quad -(1-D_2)\sinh k\lambda_2\big\}\sinh k\big] - \gamma_2^2F_3^4kD_2\sinh k, \end{aligned}$$

with

$$\begin{aligned} E_3(k) = \; & \big[\{\gamma_2^3F_3^2k\cosh k\lambda_2\sinh k\lambda_1 \\ & +\gamma_1^2F_3^2kD_1\cosh k\lambda_1\sinh k\lambda_2 \\ & -(1-D_1)\sinh k\lambda_1\sinh k\lambda_2\}E_1(k) \\ & -F_3^4\gamma_1^4kD_1\sinh(k\lambda_2)\big]/\{k\sinh(k\lambda_1)\}. \end{aligned}$$

Hence, $Q_1(x)$ and $S_1(x)$ can be determined in integral forms. The integral forms of $Q_1(x)$ and $S_1(x)$ can be evaluated once the bottom profile $f(x)$ is known. In the next section, we consider a special form for $f(x)$.

5 Special form of the bottom profile

Considering the smooth bottom profile as given by

$$f(x) = \begin{cases} \frac{1}{2}\left(1+\cos\frac{\pi x}{L}\right), & -L \le x \le L \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

and evaluating the integrals for $Q_1(x)$ and $S_1(x)$ at their singularities (the zeros of $E_2(k) = 0$), we get

$$Q_1(x) = \begin{cases} \dfrac{-2\pi^2 F_3^4 \gamma_2^2}{L^2} \displaystyle\sum_{j=0}^{2} \frac{E_1(k_j)\sinh(k_j\lambda_2)}{(\frac{\pi^2}{L^2} - k_j^2)E_2'(k_j)} \\ \qquad \times \sin k_j x \sin k_j L & \text{for } x > L \\ 0 & \text{for } x < -L \end{cases}$$

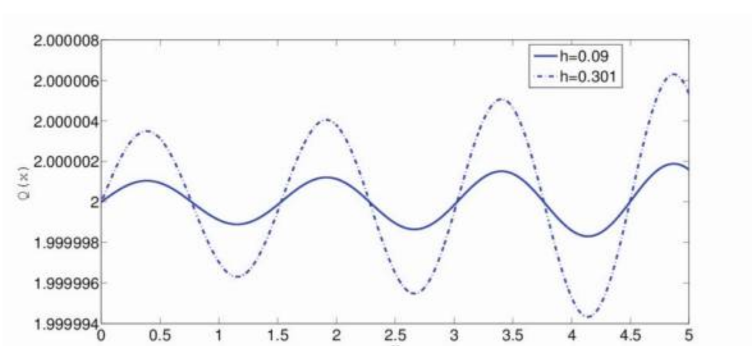
and

$$S_1(x) = \begin{cases} \dfrac{-2\pi^2 F_3^2}{L^2} \displaystyle\sum_{j=0}^{2} \frac{E_3(k_j)\sinh(k_j\lambda_2)}{(\frac{\pi^2}{L^2} - k_j^2)E_2'(k_j)} \\ \qquad \times \sin k_j x \sin k_j L & \text{for } x > L \\ 0 & \text{for } x < -L \end{cases}$$

where $E_2'(k)$ denotes the first order derivative of $E_2(k)$ with respect to k. Hence, we find that the forms of $Q_1(x)$ and $S_1(x)$ are oscillatory in nature, representing superposable waves, downstream and no wave upstream.

6 Numerical Results

The interface profiles $Q(x)$ and $S(x)$ given by the relations (12)-(13) are computed numerically and depicted in figures 1-2 respectively for different dimensionless parameters $F_1 = 0.2$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $D_1 = D_2 = 0.7$, $L = 0.5$ and for different values of the obstacle height h.



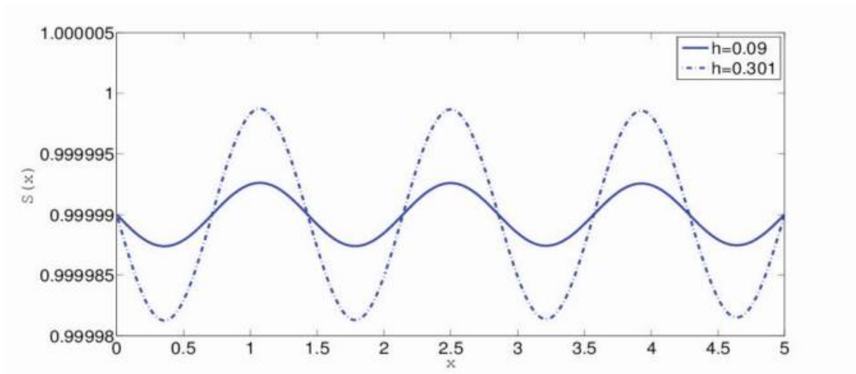


Figure 1: Wave profile of $Q(x)$ for $F_1 = 0.2$; $\gamma_1, \gamma_2 = 1$; $\lambda_1, \lambda_2 = 1$; $D_1, D_2 = 0.7$ and $L = 0.5$

Figure 2: Wave profile of $S(x)$ for $F_1 = 0.2$, $\gamma_1, \gamma_2 = 1$; $\lambda_1, \lambda_2 = 1$; $D_1, D_2 = 0.7$ and $L = 0.5$

From these figures it is clear that the forms of $Q(x)$ and $S(x)$ are oscillatory in nature and the amplitude of the profile increases when the obstacle height increases, which validate the theoretical results.

Summary and Conclusions

The fluid flow problems involving three layers of fluids in a channel having small undulation on the bottom where the uppermost fluid layer is free to the atmosphere, are investigated using two-dimensional linearized theory. Here the interfaces which varies with x (not like rigid lids) are considered as the most important practical part of the formulation. The effects of surface tension at the surfaces of separation are neglected. Perturbation analysis, in conjunction with the Fourier transform technique is used to derive the first order velocity potentials and elevations at the interfaces. The main advantage of this Fourier transform method is that we need to solve relatively easier ordinary differential equation to find the Fourier transform of the velocity potentials. From the derived results, it is clear that the amplitude of the profiles increase as the height of the obstacle increases. It is also observed that, at the interfaces we obtain superposable waves, downstream and no wave upstream and these waves are oscillatory in nature. The results developed here are expected to be helpful for a large class of multi-layered fluid flow problems in a channel with an uneven bottom.

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