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An Optimum Control of a Batch Arrival Queue with Second Optional Service and Setup time under Bernoulli Vaction Schedule

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Abstract

An $M^x/G/1$ queueing system with second optional service (SOS)is studied under N policy and Bernoulli Vacation. The system remains idle until the queue size reaches or exceeds N (≥ 1). When the queue size reaches at leastN, the server may begin his setup operation with probability s or may start the service with probability (1-s). The server provides two phases of heterogeneous services, of which, first phase of service is essential and second phase of service is optional. As soon as the first essential service (FES) of a unit is completed, the customer may leave the system with probability (1-r) or may immediately opt for SOS with probability r. Wheneverthe service of each unit is completed, the server will have the option of taking vacation(Bernoulli). Thus a customer completes his service, by undergoing FES alone, the server may take a vacation with probability $p_1$orstays idle or continue the next service to the new customer if any,with probability (1-p_1). If the customer who finishes FES proceeds to SOS then the server may take vacation at the end of SOS with probability p_2 or stays idle or continues with the FES for the new customer with probability (1-p_2) according as the system is empty or having customers in the system. The queue size distribution at a random epoch is obtained for this model using SVT and various particular cases are deduced. Further various performance measures and the optimum management policy are also derived.

Key words : Second optional service, N policy, Setup time and Bernoulli vacation.

Introduction:

This paper concerns with the steady state analysis of single server batch arrival queue with SOS service channel under Bernoulli schedule vacation and N policy. The first study of classical batch arrival queue with N policy was done by Lee and Srinivasan (1989). Later Lee etal (1994 & 1995) analysed an $M^X/G/1$ queue with N policy of multiple and single vacation respectively. In many real world production systems, setup operations are recorded in several occasions.

Recent research contribution consider thequeueing system with two phases of services. Madan (2000) introduced the concept of SOS and high lightened numerous equations of the queueing situations where all arriving customers require the main service and only some may require the subsidiary service provided by the server. Later several authors including Medhi (2003) and Choudhury (2003) analysedqueuing models with SOS

At present, most of the studies are devoted to batch arrival vacation models , under different vacation policies because of the inter disciplinary concept of Bernoulli vacation and modified service time . In Bernoulli vacation models, after each service completion the server may go for a vacation of random length V with probability p (0 ≤ p ≤ 1) or may continue to serve the next unit , if any , with probability 1-p. Otherwise, the server remains idle in the system. In most of the queueing system with Bernoulli schedule vacation, it is assumed that , the server provides , two phases of heterogeneous services one after the other to the arriving customers. In this paper we consider the Bernoulli schedule vacation for the first time along with SOS facility. One can note that the two phases of heterogeneous service of queueing model is a special case of the SOS queueing model.

Thus we have analysed the most general N policy , batch arrival queueing system with SOS channel under Bernoulli schedule vacation with or without setup operations. For this model, the steady state queue size distributions at random epoch are derived using the supplementary variable technique and various important performance measures are obtained. A cost model is proposed to obtain the optimal stationary policy under a suitable linear cost structure.Further particular cases are also derived.

Model Description:

In this paper the optimal control of $M^X/G_1\ G_2/1$ queueing system where the arrivals occur according to a compound Poisson process with arrival size of random variable X is considered under Bernoulli schedule vacation process. The server provides two types of heterogeneous services of which one is essential and the other is optional.

The server is deactivated as soon as the system becomes empty. If the queue length reaches or exceeds N, the server starts the setup operation with probability "s" whose length is a generally distributed random variable D. The customers who arrive during the idle period (or) setup period will join the queue. Immediately after the setup the server is turned on and begins to serve the first phase of essential service (FES) one ata time according to the FCFS queue discipline. After the completion of FES of a customer, the customer may leave the system with probability $(1 - r)$ (or) may opt for a SOS in an additional channel with probability r $(0 \le r \le 1)$.

If a customer leaves the system soon after the FES, the server may take a vacation (Bernoulli schedule) of random length v with probability p_1 $(0 \le p_1 \le 1)$ or may continue to serve the next customer if any with probability $(1 - p_1)$. On the other hand, if a customer finishes FES and opts for the SOS then the server takes vacation only after finishing the SOS for the customer, with probability p_2 (i.e.) thus after completing each service and sending the customer out of the system, the sever takes a vacation with probability p_j $(j = 1, 2)$ (or continue to stay in the system with probability $(1 - p_j)$. The vacation time in either case is a random variable and follows the same general distribution with finite moments. It is assumed that the server takes only a single vacation which means that whenever the vacation period of the serves ends, then he joins the system irrespective of whether there are customers waiting for the service or not .

Thus a cycle begins, when the system length reaches atleast N, and the server starts setup operation with probability s if necessary and continues with FES and SOS along with Bernoulli schedule vacation. This process continues and the cycle ends when the system becomes empty again. Thus in this model, some cycles may start directly with FES as soon as the queue length reaches or exceeds N with probability $(1 - s)$. The model under consideration is a general N-policy queueing system with heterogeneous FES and SOS facilities under Bernoulli vacation with (or) without server's setup. The random variables including service times of FES, SOS and Bernoulli vacation time are assumed to follow general law of distribution with finite moments and independent of each other. The model is denoted by $M^X/G_1\ G_2/1/V(BS)$, where V(BS) represents the Bernoulli schedule vacation and $(G_1\ G_2)$ denotes the service provided in two stages of which one is essential and the other is optional.

Notations

The arrivals occur in batches of size X whose probability distribution is given by $Pr(X = k)\ g_k$, $k = 1, 2, 3, \ldots$ and the PGF of g_k is denoted by $X(z) = \sum_{k=1}^{\infty} p_k z^k$, with mean $E(X) = X'(1)$. The services provided in two different channels by the same server are heterogeneous whose cumulative distribution function, (probability density function), {Laplace Stieltijes transform} and [remaining service time] of FEs and SOS are denoted respectively by $S_i(x)$, $(s_i(x))$, $\{S_i^*(\theta)\}$ and $[S_i^0(x)]$ for i=1, 2. $V(x)$, $v(x)$, $\{V^*(\theta)\}$ and $[V^o(x)]$; $D(x)$, $d(x)$ $\{D^*(\theta)\}$ and $[D^0(x)]$ are the corresponding notations for vacation time and setup time random variables respectively.

Let N(t) denote the number of customers present in the system at time t, the system states are denoted by C(t) = 0, 1, 2, 3 and 4 according as the server is idle, doing preparatory work, doing FES, doing SOS and on vacation respectively. The time interval between the consecutive services of each customer in this model is given by

$$
\begin{aligned}
S &= S_1 + S_2 + V && \text{with probability } p_2\, r \\
&= S_1 + S_2 && \text{with probability } (1 - p_2)\, r \\
&= S_1 + V && \text{with probability } p_1\, (1 - r) \\
&= S_1 && \text{with probability } (1 - p_1)\,(1 - r)
\end{aligned}
$$

Thus the Laplace Stieltijes transforms LST of S is

$S^*(\theta) = S_1^*(\theta)\,[(1-p_2)\,r\,S_2^*(\theta) + (1-p_1)(1-r) + V^*(\theta)\,(r\,p_2 S_2^*(\theta) + (1-r)\,p_1)]$.

And the first two moments of the random variable S are given by

$$\frac{-d}{d\theta}(S^*(\theta))\Big|_{\theta=0} = E(S);\quad \frac{-d^2}{d\theta^2}(s^*(\theta))\Big|_{\theta=0} = E(S^2).$$

It isassumed that the arrival process are independent of vacation times, setup time and service time and these random variables are also independent of each other. Further the system state probabilities are defined by using the remaining vacation time $V^o(t)$, remaining setup time $D^o(t)$, and remaining service time $S_i^o(t)$ as supplementary variables.

Thus $R_n(t) = \Pr\{N(t) = n, c(t) = 0\}\ 0 \le n \le m-1$

$D_n(x, t) = \Pr\{N(t) = n, x \le D_o(t) \le x + dt, c(t) = 1\}\quad n \ge m$

$P_{n1}(t) = \Pr\{N(t) = n, x \le S_1^o(t) \le x + dt, c(t) = 2\}\quad n \ge 1$

$P_{n2}(t) = \Pr\{N(t) = n, x \le S_2^o(t) \le x + dt, c(t) = 3\}\quad n \ge 1$

$Qn(x, t) = \Pr\{N(t) = n, x \le Vo(t) \le x + dt, c(t) = 4\}\quad n \ge 0$

where$R_n(t)$ denotes the probability that there are n customers in the system at a time t when the system is idle. $Q_n(x, t)$, $D_n(x, t)$ and $P_{ni}(x, t)$, i = 1, 2 respectively denote the probability that there are n customers in the system and the remaining vacation time, setup time and service time lie in the interval $[x, x + \Delta t]$ and $P_{ni}(0)$ i = 1, 2 ($Q_n(0)$, $D_n(0)$) denote the probability that there are n customers in the system at the terminations of service time, setup time and vacation time.

Assuming that the steady state probabilities ,

$$\lim_{t\to\infty} R_n(t)=R_n;\ \lim_{t\to\infty} Q_n(x, t)=Q_n\ ;\ \lim_{t\to\infty} D_n(x, t)=D_n(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x}(P_{ni}(x, t) = \frac{d}{dt}P_n(x)\ i = 1, 2 \text{and} \qquad \lim_{t\to\infty} \frac{\partial}{\partial t}(P_{ni}(x, t) = 0 \qquad i = 1, 2\text{exist.}$$

Now,by following the argument of Cox [1955] and observing the changes of states during the interval (t, t + Δt) at any time t, the steady state system equations are written.

$$\lambda R_0 = P_{12}(0)(1-p_2) + Q_0(0) + (1-r)\,p_{11}(1-p_1) \qquad (1)$$

$$\lambda R_n = \lambda\sum_{k=1}^{n} R_{n-k}\, g_k \qquad 1 \le n \le N-1 \qquad (2)$$

$$\frac{-d}{dx}(P_{11}(x)) = -\lambda P_{11}(x) + (1-p_2) S_1(x) P_{22}(0) + Q_1(0) S_1(x) + P_{21}(0)(1-r) S_1(x)(1-p_1) \qquad (3)$$

$$\frac{-d}{dx}(P_{n1}(x)) = -\lambda P_{n1}(x) + (1-p_2) S_1(x) P_{n+12}(0) + Q_n(0) S_1(x) + P_{n+11}(0)(1-r) S_1(x)(1-p_1) + \lambda\sum_{k=1}^{n-1} P_{n-k1}\, g_k \qquad 2 \le n \le N-1 \qquad (4)$$

$$\frac{-d}{dx}(P_{n1}(x)) = -\lambda P_{n1}(x) + (1-p_2) S_1(x) P_{n+12}(0) + Q_n(0) S_1(x) + \lambda\sum_{k=1}^{n-1} P_{n-k1}(x)\, g_k + (1-r) P_{n+11}(0)(1-p_1) S_1(x) + \lambda(1-S)\sum_{k=n-N+1}^{n} R_{n-k} \text{ of } \mu\ S_1(x) + D_n(0) S_1(x) \quad n \ge N \quad (5)$$

$$\frac{-d}{dx}(P_{12}(x)) = -\lambda P_{12}(x) + P_{11}(0) S_2(x)\, r \quad (6)$$

$$\frac{-d}{dx}(P_{n2}(x)) = -\lambda P_{n2}(x) + P_{n1}(0)\, r\, S_2(x) + \lambda\sum_{k=1}^{n-1} P_{n-k2}(x)\, g_k \qquad n \ge 2 \qquad (7)$$

$$\frac{-d}{dx}(Q_0(x)) = -\lambda Q_0(x) + p_2 P_{12}(0) V(x) + (1-r) P_{11}(0)\, p_1 V(x) \qquad (8)$$

$$\frac{-d}{dx}(Q_n(x)) = -\lambda Q_n(x) + p_2 P_{n+12}(0) V(x) + \lambda\sum_{k=1}^{n} Q_{n-k}(x)\, g_k + (1-r) P_{n+11}(0)\, p_1 V(x) \quad n \ge 1 \ (9)$$

$$\frac{-d}{dx}(D_N(x)) = -\lambda D_N(x) + \lambda \sum_{k=1}^{N} R_{N-k}\, g_k\, dx(S) \quad (10)$$

$$\frac{-d}{dx}(D_n(x)) = -\lambda D_n(x) + \lambda\, s \sum_{k=n-N+1}^{n} R_{N-k}(x)\, g_k\, dx + \lambda \sum_{k=1}^{n-N} D_{N-k}(x)\, g_k \qquad n \geq N+1 \quad (11)$$

The LST of the equations are obtained by following the definition of Laplace Stieltijes transformation and their properties. Further,the following partial probability generating functions are also defined to obtain the analytical solution. (i.e.,)

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{ni}^*(\theta) z^n; \quad P_i(z,0) = \sum_{n=1}^{\infty} P_{ni}(0) z^n (i=1,2); \quad D^*(z,\theta) = \sum_{n=m}^{\infty} D_n^*(\theta) z^n;$$

$$D(z,0) = \sum_{n=m}^{\infty} D_n(0) z^n; \quad Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta) z^n; \quad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n; \; R(z) = \sum_{n=0}^{m-1} R_n z^n$$

By consideringthe suitablepartial PGFs and algebraic techniques, we obtain,

$$R(z) = \lambda R_0 \sum_{n=0}^{N-1} \frac{\pi_n z_n}{\lambda}; \; P_2^*(z,0) = r(1 - S_2^* w_x(z)) \frac{P_1(z,0)}{(w_x(z))}; \; P_1^*(z,0) = \frac{(1-S_1^*(w_x(z)))}{S_1^*(w_x(z)))} \cdot \frac{P_1(z,0)}{(w_x(z))} \qquad Q^*(z,0) =$$

$$\frac{(1-V^*(w_x(z)))}{z}(p_1(1-r) + r S_2^*(w_x(z)\, p_2)\frac{P_1(z,0)}{(w_x(z))} \text{ and } D^*(z,0) = \frac{s(1-D^*(w_x(z)))}{(w_x(z))}(\lambda R_0 - R(z)\, w_x(z)).$$

Then total PGF is given by,

$$P(z) = P_1^*(z,0) + P_2^*(z,0) + D^*(z,0) + R(z) + Q^*(z,0)$$

$$\text{(i.e)} P(z) = \left(\frac{z-1}{z - S^*(w_x(z))}\right) S_1^*(w_x(z))((1-r) + r\, S_2^*(w_x(z))(\lambda R_0 s \left(\frac{1-D^*(w_x(z))}{w_x(z)}\right) + R(z)((1-s) + sD^*(w_x(z))) \quad (12)$$

Performance Measures

Let P_v, P_{build}, P_D and P_{busy}denotethe probability that the system is in vacation, buildup, setup and busy period respectively .Then, their corresponding system size probabilities are given by

$$P_v = \lim_{z\to 1} Q^*(z,0) = \lambda R_0 \cdot p. \lambda\, EV\, EX \frac{\left(s\,ED + \Sigma \frac{\pi_n}{\lambda}\right)}{(1-\rho)} \; ; \; P_{build} = \lim_{z\to 1} R(z) = \sum_{n=0}^{N-1} \frac{\pi_n}{\lambda} \cdot \lambda R_0$$

$$P_D = \lim_{z\to 1} D^*(z,0) = s(ED)\lambda R_0 \text{ and } P_{busy} = \lim_{z\to 1} P_1^*(z,0) + P_2^*(z,0) = \rho_1.$$

Further the value of λR_0 can be evaluated by equating the total probability to 1 and it is found that,

$$\text{(i.e.)} \lambda R_0 = \frac{\lambda(1-\rho)}{\left(\lambda s ED + \sum_{n=0}^{N-1} \pi_n\right)}$$

Decomposition Property

Now , we describe the decomposition structure. Equation (12) implies that the total PGF of the system size probabilities of the model is decomposed into the product of two random variables one of which is P_{M^X} /G_1

$$G_2/1/BV = \frac{(1-\rho)(z-1)S_1^*(w_x(z))((1-r) + r S_2^* w_x(z))}{(z - S^*(w_x(z)))}.$$

This gives the PGF of the SOS M^X/G_1 G_2/1/BV without N policy by Choudhury [2002]

and the other as
$$\psi(z) = \frac{1}{\left(s\,ED + \sum_{n=0}^{N-1} \frac{\pi_n}{\lambda}\right)} \left(\frac{s(1-D^*(w_x(z))}{w_x(z)} + (s D^*(w_x(z)) + (1-s)) \sum_{n=0}^{N-1} \frac{\pi_n z^n}{\lambda}\right)$$

gives the conditional system size distribution during the servers idle period (vacation + build up + setup).

Further it is noted that
$$\psi(z) = \frac{Q^*(z,0) + D^*(z,0) + R(z)}{Q^*(1,0) + D^*(1,0) + R(1)}$$

Mean System Size

Let L_S denote the expected system size of M^X/G_1 $G_2/1$ BV queueing system under second optional service with (or) without setup, then L_S is given by $L_S = \frac{d}{dz}(P(z))_{z=1}$

(i.e) $L_S = \mathbf{L}_{M^x/G1\ G2/1/BV} + \frac{1}{\left(\lambda s\ ED + \sum_{n=0}^{N-1} \pi_n\right)} \left(\lambda\ s\ EX\left(\frac{\lambda ED^2}{z} + ED\sum_{n=0}^{N-1}\pi_n\right) + \sum_{n=0}^{N-1} n\,\pi_n\right)$ (13) where

$\mathbf{L}_{M^x/G1\ G2/1/BV}$ gives the expected system size of $M^X/G_1G_2/1/BV$ queueingsystem without N-policy.

Optional Management Policy

In this section the main objective is to determine the optimal management policy to minimize the linear cost function while maintainingthe minimal service quantity to customers. Let C_y, C_h, C_D, C_v and C_{build}denote the cycle cost, holdingcost, setup cost, vacation cost and build up cost per unit time and $T_C(N)$ denote the average cost per unit time.

Then$T_C(N) = \frac{C_y}{E_{Cy}} + C_D P_D + C_{build}P_{build} + C_{busy}P_{busy} + C_h L_S - C_V P_V$

By substituting the values of P_D, P_{build}, P_{busy}, P_V and E_{Cy} it is found that

$T_C(N) = \frac{1}{D(N)}\left(\pi + B\sum_{n=0}^{N-1}\pi_n + C_h\sum_{n=0}^{N-1} n\,\pi_n\right) + A'$ where

$B = C_{build}(1-\rho) + C_h\lambda\ s\ EX\ ED$; $A' = C_h L_1 + C_{busy}\rho, - C_V P_V$and $D(N) = s\ \lambda\ ED + \sum_{n=0}^{N-1}\pi_n$

Thus to calculate the optimal value of $T_C(N^*)$, consider,$T_C(k+1) - T_C(k) = \frac{\pi_n}{D(k+1)\ D(k)} h_N(k)$

where$h_N(k) = (B + k\ C_h)\ \lambda\ s\ ED - C_h\Sigma\ n\ \pi_n - \overline{A}$ and $\overline{A} = \lambda\ (1-\rho)\ (C_y + C_D\ s\ ED)$

Thus the sign of $h_N(k)$ determines whether$T_C(N)$ increases (or) decreases, since $\frac{\pi_n}{D(k+1)\ D(k)} > 0$

It is observed that, $T_C(N) > T_C(N+1)$**(i.e.)** $\mathbf{N^* = \min\{k / h_N(k) > 0\}}$

Particular Cases

Case 1 : By taking s = 1, the equations (12) and (13) corresponding to the total PGF and the mean system size L_S coincides with M^X/G_1 $G_2/1/BV$ with N-policy and setup time.

(i.e.) The equation (12) becomes

$$P(z) = \left(\frac{z-1}{z - S^* w_x(z)}\right) S_1^*(w_x(z))((1-r) + r\ S_2^*(w_x(z)) \frac{(1-\rho)}{\left(ED + \Sigma\frac{\pi_n}{\lambda}\right)}\left(\left(\frac{1-D^*(w_x(z))}{w_x(z)}\right)\sum_{n=0}^{N-1}\frac{\pi_n z^n}{\lambda} D^*\ w_x(z)\right)$$

And(13) yields,$L_S = L_{M^X/G_1\ G_2/1} \frac{+\lambda^2 EX\left(\frac{ED^2}{2} + ED\sum_{n=0}^{N-1}\frac{\pi_n}{\lambda}\right) + \sum_{n=0}^{N-1} n\,\pi_n}{\lambda ED + \sum_{n=0}^{N-1}\frac{\pi_n}{\lambda}}$

Case 2 : By letting s = 0,equation (12) becomes,$P(z) = \left(\frac{z-1}{z - S^* w_x(z)}\right)(S_1^*\ w_x(z))((1-r) + r\ S_2^*(w_x(z))R(z)$

and the equation (13) results in$L_S = L_1 + \sum_{n=0}^{N-1}\frac{n\,\pi_n}{\sum_{n=0}^{N-1}\frac{\pi_n}{\lambda}}$.(i.e.) The PGF and L_S agrees with that of M^X/G_1 $G_2/1/BV$ with N policy and without setup time.Thus the model described in this paper is the generalization of the BV SOS queueing system with or without setup facility.

Conclusion:

The model discussed in this paper is among the most general queueing system with threshold policies and includes many previous works as special cases.

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