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On Unified Theorems Involving I-Function Transforms

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ABSTRACT

The present paper deals with the unified Theorem which shows relationship between originals of related functions in form of I-function transforms. Many known & new results are also obtained as particular cases.

1. Introduction

The I-function which was introduced by Saxena [16] is an extension of Fox's H-function. On specializing the parameters, I-function can be reduced to almost all the known as well as unknown special functions.

The definition of I-function given by Saxena [16] is as follows :-

$$I(z) = I^{m,n}_{p_i,q_i:r}[z]$$

$$= I^{m,n}_{p_i,q_i:r}\left[Z\middle|\begin{matrix}(a_j,\alpha_j)_{1,n}(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right]$$

$$= \frac{1}{2\pi_i}\int_C t(s)z^s\,ds \qquad \text{...(1.1)}$$

where
$$t(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}$$

p_i (i = 1, 2, ... r), q_i (i = 1, 2, ...r), m, n are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i$ (i = 1, 2, ... r) r is finite α_i, β_i, α_{ji}, β_{ji} are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers such that

$\alpha_j (b_h + v) \ne \beta_j (a_h - 1 - k)$

with all necessary conditions for existence as given by Saxena [16].

In this paper the I-function transform is defined as

$$h(s) = \bar{I}[f(x);s] = \int_0^\infty I^{M,N}_{P_i,Q_i,r}\left[sx\middle|\begin{matrix}(c_j,\gamma_j)_{1,N}(c_{ji}\gamma_{ji})_{N+1,P}\\(d_j,\delta_j)_{1,M}(d_{ji}\delta_{ji})_{M+1,Q}\end{matrix}\right]f(x)\,dx$$

where $f(x) \in A$ and A denotes the class of function for which

$$f(x) = \begin{cases}O\{x^{\xi}\}, & x\to 0\\ O\{x^{w}, e^{-w_2 x}\}, & x\to\infty\end{cases}$$

provided that existence conditions of I function satisfied.

2. Theorem 1

If $h(s) = \bar{I}[f(x);s] = \int_0^\infty I^{m,n}_{p_i,q_i;r}\left[sx\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]f(x)dx$ ...(2.1)

and $s^{-c}h(s^\sigma) = \bar{I}[g(x);s] = \int_0^\infty I^{M,N}_{P_i,Q_i;r}\left[sx\left|\begin{matrix}(c_j,\gamma_j)_{1,N}(c_{ij}\gamma_{ji})_{N+1,P_i}\\(d_j,\delta_j)_{1,M}(d_{ij}\delta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]f(x)\,dx$ ....(2.2)

then

$$g(x) = x^{c-1} = \int_0^\infty I^{m+P-N,n+Q-M}_{P+Q,q+P}\left[tx^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(d_j+c\delta_j,\sigma\delta_j)_{M+1,Q}\\(b_j,\beta_j)_{1,m}(c_j+c\gamma_j,\sigma\gamma_j)_{N+1,P}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,p}(d_j+c\delta_j,\sigma\delta_j)_{1,M}\\(b_{ji},\beta_{ji})_{m+1,q}(c_j+c\gamma_j,\sigma\gamma_j,c_j)_{1,N}\end{matrix}\right]f(t)dt \quad ...(2.3)$$

provided that the integrals involved in (2.1) to (2.3) are absolutely convergent.

Proof :

In order to prove the above theorem we use following result, which can easily be obtained by result given by Jain [9, P. 293, Eq. (6)]

$$\bar{I}\left\{x^{c-1}I^{m+P-N,n+Q-M}_{p+Q,q+P}\left[tx^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(d_{ji}+c\delta_{ji},\sigma\delta_{ji})_{M+1,Q_i}\\(b_j,\beta_j)_{1,m}(c_{ji}+c\gamma_{ji},\sigma\gamma_{ji})_{N+1,P_i}\end{matrix}\right.\right.\right.$$

$$\left.\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,p_i}(d_j+c\delta_j,\sigma\delta_j)_{1,M}\\(b_{ji},\beta_{ji})_{m+1,q_i}(c_j+c\gamma_j,\sigma\gamma_j)_{1,N}\end{matrix}\right];s\right\}$$

$$= s^c\, I^{m,n}_{p_i,q_i;r}\left[tx^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] \quad ...(2.4)$$

provided that $\sigma = 0$

$$\lambda = \sum_{j=1}^{n}A_j\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}B_j\beta_j > 0$$

$$|\arg(t)| < \frac{1}{2}\lambda\pi$$

$$\lambda = \sum_{j=1}^{N}C_j\gamma_j - \sum_{j=+1}^{P}\gamma_j + \sum_{j=1}^{M}\delta_j - \sum_{j=m+1}^{Q}D_j\delta_j > 0, |\arg(s)| < \frac{1}{2}\lambda'\pi$$

$$\sigma \max_{1\le j\le n}\operatorname{Re}\left\{\frac{a_j-1}{\alpha_j}\right\} + \max_{M+1\le j\le Q}\operatorname{Re}\left\{\frac{d_j+c\delta_j-1}{\delta_j}\right\} - \min_{1\le j\le M}\operatorname{Re}\left\{\frac{d_j}{\delta_j}\right\}$$

$$< \operatorname{Re} < \sigma \min_{1\le j\le m}\left\{\frac{b_j}{\beta_j}\right\} + \min_{N+1\le j\le P}\operatorname{Re}\left\{\frac{c_j+c\gamma_j}{\gamma_j}\right\} - \max_{1\le j<N}\operatorname{Re}\left\{\frac{c_j-1}{\gamma_j}\right\}$$

From (2.1), we get

$$s^{-c}\; h(s^\sigma) = s^{-c}\int_0^\infty I^{m,n}_{p_i,q_i;r}\left[ts^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]f(t)dt \quad ...(2.5)$$

Now by using equation (2.4), we obtain

$$s^{-c}\; h(s^\sigma) = \int_0^\infty \bar{I}\left\{x^{c-1}I^{m+P-N,n+Q-M}_{p+Q,q+P}\left[tx^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(d_{ji}+c\delta_{ji},\sigma\delta_{ji})_{M+1,Q_i}\\(b_j,\beta_j)_{1,m}(c_{ji}+c\gamma_{ji},\sigma\gamma_{ji})_{N+1,P_i}\end{matrix}\right.\right.\right.$$

$$\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,p_i}(d_j+c\delta_j,\sigma\delta_j)_{1,M}\\(b_{ji},\beta_{ji})_{m+1,q_i}(c_j+c\gamma_j,\sigma\gamma_j)_{1,N}\end{matrix}\right] f(t)\,dt$$

$$=\int_0^\infty\int_0^\infty\left\{x^{c-1}I^{m+P-N,\,n+Q-M}_{p+Q,\,q+P}\left[tx^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n}(d_{ji}+c\delta_{ji},\sigma\delta_{ji})_{M+1,Q_i}\\(b_j,\beta_j)_{1,m}(c_{ji}+c\gamma_{ji},\sigma\gamma_{ji})_{N+1,P_i}\end{matrix}\right.\right.\right.$$

$$\left.\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,p_i}(d_j+c\delta_j,\sigma\delta_j)_{1,M}\\(b_{ji},\beta_{ji})_{m+1,q_i}(c_j+c\gamma_j,\sigma\gamma_j)_{1,N}\end{matrix}\right]I^{M,N}_{P_i,Q_i;r}\left[sx\left|\begin{matrix}(c_j,\gamma_j)_{1,N}(c_{ji},\gamma_{ji})_{N+1}\\(d_j,\delta_j)_{1,M}(d_{ji},\delta_{ji})_{M+1,Q}\end{matrix}\right.\right]dx\right\}f(t)dt \quad ...(2.6)$$

On comparing (2.2) & (2.6) after changing the order of integration with all conditions mentioned with the theorem, we can easily get (2.3).

3. Special Cases

(i) If we put r = 1 in the main theorem, we get known result for H-function given by Agarwal [1] as

$$h(s)=H\{f(x);s\}=\int_0^\infty H^{m,n}_{p_i,q_i:1}\left[sx\left|\begin{matrix}(a_j\alpha_j)_{1,p_i}\\(b_j\beta_j)_{1,q_i}\end{matrix}\right.\right]f(x)dx \quad ...(3.1)$$

and

$$s^{-c}h(s^\sigma)=H\{f(x);s\}=\int_0^\infty H^{M,N}_{P,Q}\left[sx\left|\begin{matrix}(c_j\gamma_j)_{1,p}\\(d_j\delta_j)_{1,q}\end{matrix}\right.\right]g(x)dx \quad ...(3.2)$$

then $$g(x)=x^{c-1}\int_0^\infty\left\{I^{m+P-N,n+Q-M}_{p+Q,q+P}\left[tx^{-\sigma}\right]f(t)dt\right. \quad ...(3.3)$$

(ii) If we set M = m = 1, N = n = 0, P_i & p_i = 0, Q_i & q_i = 1 we obtain new result, given as

$$\text{If } h(s)=H\{f(x);s\}=\int_0^\infty \beta^{-1}(sx)^{b/\beta}e^{-x^{1/\beta}}.f(x)dx \quad ...(3.4)$$

and

$$s^{-c}h(s^\sigma)=H\{f(x);s\}==\int_0^\infty \delta^{-1}(sx)^{d/\delta}e^{-x^{1/\delta}}.g(x)dx \quad ...(3.5)$$

then

$$g(x)=x^{c-1}\int_0^\infty\left\{H^{1,0}_{1,1}\left[tx^{-\sigma}\left|\begin{matrix}(d,\sigma\delta)\\(b,\beta)\end{matrix}\right.\right]f(t)dt\right. \quad ...(3.6)$$

(iii) By using relation of H function with general Hurwitz-Lerch zeta function [2, P.27, eq. 11)]

$$\phi(z,p,\eta)=\sum_{n=0}^\infty\frac{1}{(\eta+n)^p}Z^n=\overline{H}^{1,2}_{2,2}\left[-z\left|\begin{matrix}(0,1,1)(1-\eta,1;p)\\(0,1)(-\eta,1;p)\end{matrix}\right.\right] \quad ...(3.7)$$

We get

$$\text{If } h(s)=\phi\{f(x);p,\eta:s\}=\int_0^\infty\phi(-sx,p,\eta)f(x)dx \quad ...(3.8)$$

and

$$s^{-c}h(s^{\sigma}) = \phi\{g(x); p', \eta'; s\} = \int_0^{\infty} \phi(-sx, p', \eta')\, g(x)\, dx \qquad ...(3.9)$$

then

$$g(x) = x^{c-1}\int_0^{\infty} \overline{H}_{4,4}^{1,3}\left[tx^{-\sigma} \middle| \begin{matrix} (0,1;1)(1-\eta,1,p)(-\eta'+c,\sigma;p')(c,\sigma) \\ (0,1)(-\eta,1,p)(1-\eta'+c,\sigma',p') \end{matrix} \right] f(t)dt \qquad ...(3.10)$$

provided all integrals are absolutely convergent.

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