HAUSDORFF EXTENSIONS OF SPACES IN INTUITIONISTIC FUZZY \( S^* \) CENTRED SYSTEMS

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Abstract

In this paper, the concepts of maximal intuitionistic fuzzy \( S^* \) centred systems and Hausdorff extensions of spaces in intuitionistic fuzzy \( S^* \) centred system are studied as in [9] and their properties are discussed.

Key words

Intuitionistic fuzzy \( S^* \) structure space \((S, S^*)\), intuitionistic fuzzy \( S^* \) centred system, a maximal intuitionistic fuzzy \( S^* \) centred system, intuitionistic fuzzy \( S^* \) \( H \)-closed, intuitionistic fuzzy \( S^* \) compact, intuitionistic fuzzy \( S^* \) completely regular space, intuitionistic fuzzy \( S^* \) maximal compact extension \( \beta(S) \).

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [13], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of “Intuitionistic fuzzy sets” was first published by Atanassov [1] and many works by the same author and his colleagues appeared in the literature [2-4]. Later this concept was generalized to “Intuitionistic \( L \)-fuzzy sets” by Atanassov and stoeva [5]. An introduction to intuitionistic fuzzy topological space was introduced by Dogan Coker [6]. Fuzzy continuity in intuitionistic fuzzy topological spaces was introduced by Gurcay. H, Coker, D., and Haydar, Es. A., [8]. The method of centered system in the theory of topological spaces was introduced by Iliadis. S, Fomin.S [9], in order to study, as far as possible from one point of view, a number of problems in the theory of extension of topological spaces, like compactness and \( H \)-closure, as well as a newer range of ideas concerned principally with the work of Gleason [7] and Ponomarev [10]. The concept of intuitionistic fuzzy compactness was found in [6] and intuitionistic fuzzy Hausdorff space was found in [6]. Uma, Roja and Balasubramanian [12] extended the method of centred systems to fuzzy topological spaces. In this chapter, the concepts of maximal intuitionistic fuzzy \( S^* \) centred systems, intuitionistic fuzzy \( S^* \) Hausdorff extensions of spaces in intuitionistic fuzzy \( S^* \) centred system, the realization of an arbitrary intuitionistic fuzzy \( S^* \) extension of a space of intuitionistic fuzzy \( S^* \) centered systems, intuitionistic fuzzy
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$S$-Hausdorff extensions of types $\sigma$ and $\tau$, maximality of intuitionistic fuzzy $S$-spaces $\sigma(S)$ and $\tau(S)$ and classes of intuitionistic fuzzy $S$-Hausdorff extensions of $S$ are studied.

2. Preliminaries

Definition 2.1[3] Let $X$ be a non empty fixed set. An intuitionistic fuzzy set (IFS for short) $A$ is an object having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ where the functions $\mu_A : X \rightarrow I$ and $\gamma_A : X \rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Remark 2.1[6] For the sake of simplicity, we shall use the symbol $A = \{ x, \mu_A, \gamma_A \}$.

Definition 2.2[6] Let $X$ be a non empty set and let $\{ A_i : i \in J \}$ be an arbitrary family of IFSs in $X$. Then

(a) $\bigcap A_i = \{ (x, \wedge \mu_A(x), \vee \gamma_A(x)) : x \in X \}$;

(b) $\bigcup A_i = \{ (x, \vee \mu_A(x), \wedge \gamma_A(x)) : x \in X \}$.

Definition 2.3[6] Let $X$ be a non empty fixed set. Then, $0_- = \{ (x, 0,1) : x \in X \}$ and $1_- = \{ (x, 1,0) : x \in X \}$.

Definition 2.4[6] Let $X$ and $Y$ be two non empty fixed sets and $f : X \rightarrow Y$ be a function. Then

(a) If $B = \{ (y, \mu_B(y), \gamma_B(y)) : y \in Y \}$ is an IFS in $Y$, then the pre image of $B$ under $f$, denoted by $f^{-1}(B)$, is the IFS in $X$ defined by $f^{-1}(B) = \{ (x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)) : x \in X \}$.

(b) If $A = \{ (x, \lambda_A(x), \nu_A(x)) : x \in X \}$ is an IFS in $X$, then the image of $A$ under $f$, denoted by $f(A)$, is the IFS in $Y$ defined by $f(A) = \{ (y, f(\lambda_A)(y), (1-f)(1-\nu_A))(y)) : y \in Y \}$ where,

\[
 f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq 0 \\ 0, & \text{otherwise,} \end{cases}
 \]

\[
 (1-f(1-\nu_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq 0 \\ 1, & \text{otherwise.} \end{cases}
 \]

for the IFS $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$.

Definition 2.5[6] Let $X$ be a non empty set. An intuitionistic fuzzy topology (IFT for short) on a non empty set $X$ is a family $\tau$ of intuitionistic fuzzy sets (IFSs for short) in $X$ satisfying the following axioms: (T1) $0, 1 \in \tau$, (T2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$, (T3) $\bigcup G_i \in \tau$ for any arbitrary family $\{ G_i : i \in J \}$.

For the sake of simplicity, we shall use the symbol $A = \{ x, \mu_A, \gamma_A \}$.
In this case the pair \((X, \tau)\) is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in \(\tau\) is known as an intuitionistic fuzzy open set (IFOS for short) in \(X\).

**Definition 2.6**[6] Let \(X\) be a non empty set. The complement \(\overline{A}\) of an IFOS \(A\) in an IFTS \((X, \tau)\) is called an intuitionistic fuzzy closed set (IFCS for short) in \(X\).

**Definition 2.7**[6] Let \((X, \tau)\) be an IFTS and \(A = \{x, \mu_A, \gamma_A\}\) be an IFS in \(X\). Then the fuzzy interior and fuzzy closure of \(A\) are defined by
\[
\text{cl}(A) = \bigcap \{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\},
\]
\[
\text{int}(A) = \bigcup \{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}.
\]

**Remark 2.2**[6] Let \((X, \tau)\) be an IFTS. \(\text{cl}(A)\) is an IFCS and \(\text{int}(A)\) is an IFOS in \(X\), and (a) \(A\) is an IFS in \(X\) iff \(\text{cl}(A) = A\); (b) \(A\) is an IFOS in \(X\) iff \(\text{int}(A) = A\).

**Proposition 2.1**[6] Let \((X, \tau)\) be an IFTS. For any IFS \(A\) in \((X, \tau)\), we have
(a) \(\text{cl}(\overline{A}) = \text{int}(A)\), (b) \(\text{int}(A) = \text{cl}(\overline{A})\).

**Definition 2.8**[6] Let \((X, \tau)\) and \((Y, \varphi)\) be two IFTSs and let \(f : X \to Y\) be a function. Then \(f\) is said to be fuzzy continuous iff the pre image of each IFS in \(\varphi\) is an IFS in \(\tau\).

**Definition 2.9**[6] Let \((X, \tau)\) and \((Y, \varphi)\) be two IFTSs and let \(f : X \to Y\) be a function. Then \(f\) is said to be fuzzy open(resp. closed) iff the image of each IFS in \(\tau\) (resp. \(1-\tau\)) is an IFS in \(\varphi\) (resp. \(1-\varphi\)).

**Definition 2.10**[11] A fuzzy topological space \((X, T)\) is called fuzzy Hausdorff space iff for any two distinct fuzzy points \(p, q \in X\), there exists \(U, V \in T\) such that \(U \cap V = \emptyset\) with \(p \in U\) and \(q \in V\).

**Definition 2.11**[12] Let \(R\) be a fuzzy Hausdorff space. A system \(p = \{U_\alpha\}\) of fuzzy open sets of \(R\) is called fuzzy centred if any finite collection of the sets of the system has a non empty intersection. The system \(p\) is called a maximal fuzzy centred system or a fuzzy end if it cannot be included in any larger fuzzy centred system of fuzzy open sets.

**Definition 2.12**[9] A Hausdorff space \(\hat{\delta}(\hat{R})\) is called an extension of a Hausdorff space \(R\) if \(R\) is contained in \(\hat{\delta}(\hat{R})\) as an everywhere dense subset. \(R\) is called \(H\)-closed if every extension coincides with \(R\) itself. An extension \(\hat{\delta}(\hat{R})\) is called \(H\)-closed and compact if \(\hat{\delta}(\hat{R})\) is compact.

### 3.1 **Intuitionistic fuzzy S* Hausdorff Extensions of Spaces**

In this section, three criteria for intuitionistic fuzzy \(H\)-closure are studied and interesting properties and characterizations are established.

**Definition 3.1** Let \(X\) be a non empty set. Let \(S\) be a collection of all intuitionistic fuzzy sets of \(X\). An intuitionistic fuzzy \(S^*\) structure on \(S\) is a collection \(S^*\) of subsets of \(S\) having the following properties
(a) \(\emptyset\) and \(S\) are in \(S^*\).
(b) The union of the elements of any sub collection of \(S^*\) is in \(S^*\).
(c) The intersection of the elements of any finite sub collection of \(S^*\) is in \(S^*\).
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The collection $S$ together with the structure $S^*$ is called intuitionistic fuzzy $S^*$ structure space. A subset $U$ of $S$ is an intuitionistic fuzzy $S^*$ open set if $U \in S^*$. Its complement is said to be an intuitionistic fuzzy $S^*$ closed set.

**Definition 3.2** Let $A$ be a member of $S$. An intuitionistic fuzzy $S^*$ open set $U$ in $(S, S^*)$ is said to be an intuitionistic fuzzy $S^*$ neighbourhood of $A$ if $A \in G \subseteq U$ for some intuitionistic fuzzy $S^*$ open set $G$ in $(S, S^*)$.

**Definition 3.3** Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ structure space and $A = \{x, \mu_A, \gamma_A\}$ be an intuitionistic fuzzy set in $X$. Then the intuitionistic fuzzy $S^*$ closure and intuitionistic fuzzy $S^*$ interior of $A$ are respectively defined and denoted by

$$IF S^* \text{cl}(A) = \bigcap\{ K : K \text{ is an intuitionistic fuzzy } S^* \text{ closed sets in } S \text{ and } A \subseteq K \},$$

$$IF S^* \text{int}(A) = \bigcup\{ G : G \text{ is an intuitionistic fuzzy } S^* \text{ open sets in } S \text{ and } G \subseteq A \}.$$ 

**Definition 3.4** The ordered pair $(S, S^*)$ is called an intuitionistic fuzzy $S^*$ Hausdorff space if for each pair $A_1, A_2$ of disjoint members of $S$, there exist disjoint intuitionistic fuzzy $S^*$ open sets $U_1$ and $U_2$ such that $A_1 \subset U_1$ and $A_2 \subset U_2$.

**Definition 3.5** Let $(S_1, S_1^*)$ and $(S_2, S_2^*)$ be any two intuitionistic fuzzy $S^*$ structure spaces and let $f : (S_1, S_1^*) \rightarrow (S_2, S_2^*)$ be a function. Then $f$ is said to be an intuitionistic fuzzy $S^*$ continuous if the pre image of each intuitionistic fuzzy $S_2^*$ open set in $(S_2, S_2^*)$ is an intuitionistic fuzzy $S_1^*$ open set in $(S_1, S_1^*)$.

**Definition 3.6** Let $(S_1, S_1^*)$ and $(S_2, S_2^*)$ be two intuitionistic fuzzy $S^*$ structure spaces and let $f : (S_1, S_1^*) \rightarrow (S_2, S_2^*)$ be a bijection function. If both the function $f$ and the inverse function $f^{-1} : (S_2, S_2^*) \rightarrow (S_1, S_1^*)$ are intuitionistic fuzzy $S^*$ continuous, then $f$ is called a intuitionistic fuzzy $S^*$ homeomorphism.

**Definition 3.7** Let $f$ be a function from an intuitionistic fuzzy $S^*$ structure space $(S_1, S_1^*)$ into an intuitionistic fuzzy $S^*$ structure space $(S_2, S_2^*)$ with $f(A_1) = A_2$ where $A_1 \in (S_1, S_1^*)$ and $A_2 \in (S_2, S_2^*)$. Then $f$ is called an intuitionistic fuzzy $S^*$ $\theta$-continuous at $A_1$ if for every neighbourhood $O_{A_2}$ of $A_2$ there exists a neighbourhood $O_{A_1}$ of $A_1$ such that $f(IFS^*cl(O_{A_1}) \subseteq IFS^*cl(O_{A_2})$. The function is called intuitionistic fuzzy $S^*$ $\theta$-continuous if it is intuitionistic fuzzy $S^*$ $\theta$-continuous at every member of $S_1$. A function is called an intuitionistic fuzzy $S^*$ $\theta$-homeomorphism if it is intuitionistic fuzzy $S^*$ one to one and intuitionistic fuzzy $S^*$ $\theta$-continuous in both directions.

**Definition 3.8** Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ Hausdorff space. A system $p = \{U_\alpha : \alpha = 1, 2, 3, ..., n\}$ of intuitionistic fuzzy $S^*$ open sets is called an intuitionistic fuzzy $S^*$ centred system if any finite collection of the sets of the system has a non empty intersection. The intuitionistic fuzzy $S^*$ centred system $p$ is called a maximal
intuitionistic fuzzy $S^*$ centred system or an intuitionistic fuzzy $S^*$ end if it cannot be included in any larger intuitionistic fuzzy $S^*$ centred system of intuitionistic fuzzy $S^*$ open sets.

Note 3.1 Throughout this chapter let $U_\alpha : \alpha = 1, 2, 3, ..., n$ be an intuitionistic fuzzy $S^*$ open set in $(S, S^*)$.

Proposition 3.1 Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ Hausdorff space and $p = \{U_\alpha \}$ is an intuitionistic fuzzy $S^*$ centred system in $(S, S^*)$. Then the following properties hold

a. If $U_i \in p$ ($i = 1, 2, ..., n$), then $\bigcap_{i=1}^n U_i \in p$.

b. If \( \emptyset \neq U \subseteq H \), $U \in p$ and $H$ is intuitionistic fuzzy $S^*$ open set, then $H \in p$.

c. If $H$ is an intuitionistic fuzzy $S^*$ open set, then $H \notin p$ iff there exists $U \in p$ such that $U \cap H = \emptyset$.

d. If $U_1 \cup U_2 = U_3 \in p$, $U_1$ and $U_2$ are intuitionistic fuzzy $S^*$ open sets and $U_1 \cap U_2 = \emptyset$, then either $U_1 \in p$ or $U_2 \in p$.

e. If $IFS^*cl(U) = S$, then $\emptyset \neq U \in p$ for any intuitionistic fuzzy $S^*$ end $p$.

Definition 3.9 Let $\Theta(S)$ denote the collection of all intuitionistic fuzzy $S^*$ ends belonging to $S$. An intuitionistic fuzzy $S^*$ topology is introduced into $\Theta(S)$ in the following way. Let $O_U$ be the set of all intuitionistic fuzzy $S^*$ ends that contains $U$ as an element, where $U$ is an intuitionistic fuzzy $S^*$ open set of $S$. Therefore $O_U$ is an intuitionistic fuzzy $S^*$ neighbourhood of each intuitionistic fuzzy $S^*$ end contained in $O_U$.

Definition 3.10 A subset $A$ of an intuitionistic fuzzy $S^*$ structure space $(S, S^*)$ is said to be an everywhere intuitionistic fuzzy $S^*$ dense subset in $(S, S^*)$ if $IFS^*cl(A) = S$.

A subset of an intuitionistic fuzzy $S^*$ structure space $(S, S^*)$ is said to be a nowhere intuitionistic fuzzy $S^*$ dense subset in $(S, S^*)$ if $X \setminus \overline{A}$ is everywhere intuitionistic fuzzy $S^*$ dense subset.

Definition 3.11 Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ structure space and $Y$ be an intuitionistic fuzzy $S^*$ open set in $(S, S^*)$. Then the intuitionistic fuzzy $S^*$ relative topology $T_Y = \{G \cap Y : G \in S^*\}$ is called the intuitionistic fuzzy $S^*$ relative (or induced or subspace) topology on $Y$. The ordered pair $(Y, T_Y)$ is called an intuitionistic fuzzy $S^*$ subspace of the intuitionistic fuzzy $S^*$ space $(S, S^*)$.

Definition 3.12 Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ structure space. Then

(a) If a family $\{U_\alpha : i \in \Lambda\}$ of intuitionistic fuzzy $S^*$ open sets in $(S, S^*)$ satisfies the condition $S = \bigcup\{U_\alpha : i \in \Lambda\}$, then it is called an intuitionistic fuzzy $S^*$ open cover of $S$. A finite subfamily of the intuitionistic fuzzy $S^*$ open cover $\{U_\alpha : i \in \Lambda\}$ of $S$, which is also an intuitionistic fuzzy $S^*$ open cover of $S$, is called a finite subcover of $\{U_\alpha : i \in \Lambda\}$.
(b) An intuitionistic fuzzy $S^*$ structure space $(S, S^*)$ is called intuitionistic fuzzy $S^*$ compact iff every intuitionistic fuzzy $S^*$ open cover of $S$ has a finite subcover.

**Definition 3.13** An intuitionistic fuzzy $S^*$ Hausdorff space $\delta(S)$ is called an extension of an intuitionistic fuzzy $S^*$ Hausdorff space $S$ if $S$ is contained in $\delta(S)$ as an everywhere intuitionistic fuzzy $S^*$ dense subset. $S$ is called an intuitionistic fuzzy $S^*H$-closed if every extension coincides with $S$ itself. An extension $\delta(S)$ is called an intuitionistic fuzzy $S^*H$-closed if $\delta(S)$ is an intuitionistic fuzzy $S^*H$-closed, and intuitionistic fuzzy $S^*$ compact if $\delta(S)$ is an intuitionistic fuzzy $S^*$ compact.

**Definition 3.14** Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ structure space. A system $\mathcal{B}$ of intuitionistic fuzzy $S^*$ open sets of an intuitionistic fuzzy $S^*$ structure space $S$ is called an intuitionistic fuzzy $S^*$ base (or basis) for $(S, S^*)$ if each member of $(S, S^*)$ is a union of members of $\mathcal{B}$. A member of $\mathcal{B}$ is called an intuitionistic fuzzy $S^*$ basic open set.

**Definition 3.15** Let $(S, S^*)$ be an intuitionistic fuzzy $S^*$ structure space. A system of intuitionistic fuzzy $S^*$ open sets of an intuitionistic fuzzy $S^*$ structure space $S$ is called an intuitionistic fuzzy $S^*$ sub-base if it together with all possible finite intersections of members of the system forms a base of $S$.

**Lemma 3.1** An intuitionistic fuzzy $S^*$ structure space $S$ is intuitionistic fuzzy $S^*H$-closed if and only if any intuitionistic fuzzy $S^*$ centered system $\{U_\alpha\}$ of intuitionistic fuzzy $S^*$ open sets of $S$ satisfies the condition

$$\bigcap_\alpha IFS^*\text{cl}(U_\alpha) \neq \emptyset$$

**Lemma 3.2** An intuitionistic fuzzy $S^*$ structure space $S$ is intuitionistic fuzzy $S^*H$-closed if and only if any maximal intuitionistic fuzzy $S^*$ centered system $\{U_\alpha\}$ of intuitionistic fuzzy $S^*$ open sets of $S$ contains all the intuitionistic fuzzy $S^*$ neighbourhoods of some member.

The proof of this lemma follows easily from Lemma 3.1.

**Lemma 3.3** The intuitionistic fuzzy $S^*$ structure space $S$ is intuitionistic fuzzy $S^*H$-closed if and only if from any intuitionistic fuzzy $S^*$ cover $\{U_\alpha\}$ of $S$ a finite subsystem $U_i (i = 1, 2, 3, \ldots, n)$ may be chosen such that

$$\bigcup_{i=1}^n IFS^*\text{cl}(U_i) = S.$$ 

The proof of this lemma follows easily from Lemma 3.1.

**4. THE REALIZATION OF AN ARBITRARY INTUITIONISTIC FUZZY $S^*$ EXTENSION OF A SPACE OF INTUITIONISTIC FUZZY $S^*$ CENTERED SYSTEMS.**

In this section, intuitionistic fuzzy $S^*$ extensions of types $\sigma$ and $\tau$, the intuitionistic fuzzy $S^*$ structure space $\sigma(S)$, intuitionistic fuzzy $S^*$ structure space $\tau(S)$, maximality of the intuitionistic fuzzy $S^*$ structure spaces $\sigma(S)$ and $\tau(S)$ and classes of intuitionistic fuzzy $S^*$ Hausdorff extensions of $S$ are introduced. And some interesting properties are discussed.
Let \( \{q\} \) be a collection of intuitionistic fuzzy \( S^* \) centered (not necessarily maximal) systems of intuitionistic fuzzy \( S^* \) open sets of \( S \). An intuitionistic fuzzy \( S^* \) topology may be defined on this collection, similar to that introduced above in \( \delta(S) \). For if \( U \) is an intuitionistic fuzzy \( S^* \) open set of \( S \), let \( O_U \) denote the collection of all intuitionistic fuzzy \( S^* \) centered systems \( q \in \{q\} \) containing \( U \) as an element. All sets of the form \( O_U \) form a subbase.

Let \( \delta(S) \) be an arbitrary intuitionistic fuzzy \( S^* \) extension of \( S \). Every member \( A \in \delta(S) \) is of type \( \Theta \)-homeomorphic to \( \delta(S) \). Thus, every extension of an arbitrary intuitionistic fuzzy \( S^* \) Hausdorff space \( S \) can be realized as an intuitionistic fuzzy \( S^* \) structure space of centered systems of intuitionistic fuzzy \( S^* \) open sets of \( S \) with an appropriately chosen topology.

Definition 4.1

An intuitionistic fuzzy \( S^* \) extension \( \delta(S) \) is of type \( \sigma \) if the function \( i \) (one-to-one correspondence between the members of \( \delta(S) \) and \( \delta_\sigma(S) \)) is an intuitionistic fuzzy \( S^* \) \( \Theta \)-homeomorphism.

Definition 4.2

An intuitionistic fuzzy \( S^* \) extension \( \delta(S) \) is of type \( \tau \) if the set \( \delta(S) \setminus S \) is discrete in the intuitionistic fuzzy \( S^* \) relative topology.

Proposition 4.1

Every intuitionistic fuzzy \( S^* \) extension of \( S \) is a relict fuzzy \( S^* \) \( \Theta \)-homeomorphic to some extension of type \( \sigma \) of the same space.

The proof follows from the fact the intuitionistic fuzzy \( S^* \) extension \( \delta_\sigma(S) \) in Lemma 4.1 is of type \( \sigma \).

Now let \( \delta(S) \) be any intuitionistic fuzzy \( S^* \) extension. Let \( \delta_\tau(S) \) denote the intuitionistic fuzzy \( S^* \) structure space obtained as follows. The members of \( \delta_\tau(S) \) are those of \( \delta(S) \). The intuitionistic fuzzy \( S^* \) neighbourhoods of members \( A \in S \) are same as in \( S \), but for members \( A \in \delta(S) \setminus S \), the intuitionistic fuzzy \( S^* \) neighbourhoods are obtained from those of \( A \) in \( \delta(S) \) by rejecting the set \( \delta(S) \setminus S \cup A \). Clearly \( \delta_\tau(S) \) an intuitionistic fuzzy \( S^* \) Hausdorff space.

Definition 4.3
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Let \((S_1, S_1^*)\) and \((S_2, S_2^*)\) be two intuitionistic fuzzy \(S^*\) structure spaces. An intuitionistic fuzzy \(S^*\) structure space \((S_1, S_1^*)\) is said to be topologically embedded in another intuitionistic fuzzy \(S^*\) structure space \((S_2, S_2^*)\) if \((S_1, S_1^*)\) is an intuitionistic fuzzy \(S^*\) homeomorphism to an intuitionistic fuzzy \(S^*\) subspace of \((S_2, S_2^*)\).

**Lemma 4.2** For any intuitionistic fuzzy \(S^*\) extension \(\delta(S)\), the intuitionistic fuzzy \(S^*\) structure space \(\delta_\tau(S)\) is an intuitionistic fuzzy \(S^*\) extension of \(S\), intuitionistic fuzzy \(S^*\) \(\Theta\)-homeomorphic to \(\delta(S)\) and of type \(\tau\).

**Note 4.1** From Lemma 4.2 each intuitionistic fuzzy \(S^*\) extension \(\delta(S)\) of \(S\) is associated with intuitionistic fuzzy \(S^*\) extensions \(\delta_\sigma(S)\) and \(\delta_\tau(S)\), of types \(\sigma\) and \(\tau\), respectively and intuitionistic fuzzy \(S^*\) \(\Theta\)-homeomorphic to each other and also intuitionistic fuzzy \(S^*\) \(\Theta\)-homeomorphic to the original intuitionistic fuzzy \(S^*\) extension \(\delta(S)\).

Let \(G\) be a intuitionistic fuzzy \(S^*\) base of intuitionistic fuzzy \(S^*\) open sets in an intuitionistic fuzzy \(S^*\) structure space \(S\) and \(\sigma_{G}(S)\), the intuitionistic fuzzy \(S^*\) structure space whose elements are, firstly, the members of \(S\) itself and secondly, all the maximal intuitionistic fuzzy \(S^*\) centred systems \(\{U_{\alpha}\}\) consisting of intuitionistic fuzzy \(S^*\) open sets belonging to \(G\), none of which contains as a subsystem all the intuitionistic fuzzy \(S^*\) neighbourhoods of any intuitionistic fuzzy \(S^*\) open set of \(S\) belonging to \(G\) (Clearly this condition is equivalent to the following: \(\bigcap_{\alpha} IF^{S^*} cl(U_{\alpha}) = \phi\)).

An intuitionistic fuzzy \(S^*\) topology is defined in \(\sigma_{G}(S)\) as follows. If \(U \in G\), then \(O_U\) denotes the set of all \(A \in U\) and all maximal intuitionistic fuzzy \(S^*\) centred system in \(\sigma_{G}(S)\) that contain \(U\) as an element. Since in \(\sigma_{G}(S)\) each member \(A \in S\) can be replaced by the intuitionistic fuzzy \(S^*\) centred system of all its intuitionistic fuzzy \(S^*\) neighbourhoods belonging to \(G\) (with the intuitionistic fuzzy \(S^*\) topologization: \(\{U_{\alpha}\} \in O_U\) if \(U \in \{U_{\alpha}\}\)). It is clear that each \(\sigma_{G}(S)\) is an intuitionistic fuzzy \(S^*\) Hausdorff extension of type \(\sigma\) of the original intuitionistic fuzzy \(S^*\) structure space \(S\).

The intuitionistic fuzzy \(S^*\) structure space \(\sigma_{G}(S)\) may also be obtained by the following equivalent method. An intuitionistic fuzzy \(S^*\) centred system \(\{U_{\alpha}\}\) of intuitionistic fuzzy \(S^*\) open sets of \(G\) is called an intuitionistic fuzzy \(S^*\) Hausdorff system if for every \(B \in S\) not belonging to \(U \in \{U_{\alpha}\}\) there exists a \(U' \in \{U_{\alpha}\}\) such that \(B \not\in IF^{S^*} cl(U')\). A maximal intuitionistic fuzzy \(S^*\) Hausdorff system (that is, one which cannot be extended while remaining an intuitionistic fuzzy \(S^*\) centred system and an intuitionistic fuzzy \(S^*\) Hausdorff system) is called an intuitionistic fuzzy \(S^*\) Hausdorff end. Then the set of all intuitionistic fuzzy \(S^*\) Hausdorff ends with the intuitionistic fuzzy \(S^*\) topology described in Section 4 is again \(\sigma_{G}(S)\). To establish this it is sufficient to note firstly that the set of all intuitionistic fuzzy \(S^*\) neighbourhoods of \(G\) of an arbitrary fixed intuitionistic fuzzy \(S^*\) open set \(A\)
of an intuitionistic fuzzy $S^*$ Hausdorff end, and secondly that every maximal intuitionistic fuzzy $S^*$ centred system not containing all the intuitionistic fuzzy $S^*$ neighbourhoods in $\mathcal{G}$ of some member is automatically intuitionistic fuzzy $S^*$ Hausdorff end. As the base it may be taken, in particular, the set of all intuitionistic fuzzy $S^*$ open sets of $S$. An intuitionistic fuzzy $S^*$ structure space $\sigma_\varphi(S)$ associated with this base will simply be denoted by $\sigma(S)$.

**Proposition 4.2** An intuitionistic fuzzy $S^*$ extension $\sigma(S)$ is an intuitionistic fuzzy $S^*$ $H$-closed extension of $S$.

**Note 4.2** Let $\mathcal{G}$ be any intuitionistic fuzzy $S^*$ base of $S$. If $U \in \mathcal{G}$ then, $S \setminus IFS^*cl(U) \in \mathcal{G}$. Then $\mathcal{G}$ is called intuitionistic fuzzy $S^*$ algebraically closed.

**Remark 4.1**

If $\mathcal{G}$ is an intuitionistic fuzzy $S^*$ algebraically closed base of $S$, then $\sigma_\varphi(S)$ is an intuitionistic fuzzy $S^*$ $H$-closed extension of $S$.

The proof is essentially the same as that of Proposition 4.2.

**Note 4.3** Each intuitionistic fuzzy $S^*$ extension $\delta(S)$ is associated with intuitionistic fuzzy $S^*$ $\theta$-homeomorphic extension $\sigma_\tau(S)$ of type $\tau$, the intuitionistic fuzzy $S^*$ structure space $\delta_\tau(S)$ which is associated with $\sigma(S)$ is denoted by $\tau(S)$ and is called an intuitionistic fuzzy $S^*$ Katetov extension of $S$.

**Lemma 4.3** An intuitionistic fuzzy $S^*$ $\theta$-continuous image of an intuitionistic fuzzy $S^*$ $H$-closed space is an intuitionistic fuzzy $S^*$ $H$-closed extension of $S$.

**Remark 4.2** The intuitionistic fuzzy $S^*$ structure space $\tau(S)$ is an intuitionistic fuzzy $S^*$ $H$-closed extension of $S$.

The proof follows from Proposition 4.2 and Lemma 4.3.

**Note 4.4** An intuitionistic fuzzy $S^*$ structure space $\tau(S)$ has the following maximal properties.

**Proposition 4.3** If $\delta(S)$ is any (not necessarily intuitionistic fuzzy $S^*$ $H$-closed) intuitionistic fuzzy $S^*$ extension of $S$, then there exists a subset $\tau_\delta(S) \subseteq \tau(S)$, containing $S$ and an intuitionistic fuzzy $S^*$ continuous function $\delta_\tau$ of this subset onto $\delta(S)$ such that $\delta_\tau(A) = A$, where $A \in S$. Here if $\delta(S)$ is an intuitionistic fuzzy $S^*$ $H$-closed extension, it may be assumed that $\tau_\delta(S) = \tau(S)$.

**Remark 4.3.**

$\tau_\delta(S)$ denotes the intuitionistic fuzzy $S^*$ structure space obtained from $\tau_\delta(S)$ by the procedure described in Section 4. It is easy to see that $\tau_\delta(S)$ is an intuitionistic fuzzy $S^*$ $\theta$-homeomorphic to a subset of the intuitionistic fuzzy $S^*$ extension $\sigma(S)$. As $\tau_\delta(S)$ is an intuitionistic fuzzy $S^*$ $\theta$-homeomorphic to $\tau_\delta(S)$. Proposition 4.3 holds if $\tau(S)$ is replaced by $\sigma(S)$ and intuitionistic fuzzy $S^*$ continuity by intuitionistic fuzzy $S^*$ $\theta$-continuity.

**Remark 4.4** An intuitionistic fuzzy $S^*$ structure space $\sigma(S)$ can be mapped intuitionistic fuzzy $S^*$ $\theta$-continuity onto any intuitionistic fuzzy $S^*$ $H$-closed extension of $S$ in such a way that the members of $S$ remain fixed.

**Lemma 4.4** An intuitionistic fuzzy $S^*$ open set $\tau_\delta(S)$ is the largest subset of $\tau(S)$ that can be continuously mapped onto $\delta(S)$ in such a way that the members of $S$ remain fixed; in other words, if a set $\tau_\delta(S)$ is continuously mapped onto $\delta(S)$ in such a way that the members of $S$ remain fixed, then $\tau_\delta(S) \subseteq \tau_\delta(S)$.
Note 4.5 Thus, all intuitionistic fuzzy $S^*$ extensions of $S$ fall into classes, where $\delta(S)$ and $\delta'(S)$ are in the same class if and only if $\tau_\delta(S) = \tau_\delta'(S)$. All intuitionistic fuzzy $S^*$ $H$-closed extensions belong to the same class, by Lemma 4.3 contains only intuitionistic fuzzy $S^*$ $H$-closed extensions.

Lemma 4.5 If intuitionistic fuzzy $S^*$ extensions $\delta(S)$ and $\gamma(S)$ are intuitionistic fuzzy $S^*$ $\Theta$-homeomorphic, then they belong to the same class, that is $\tau_\delta(S) = \tau_\gamma(S)$.

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