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PREPOSTEROUS LOOM ON IRRATIONALITY OF NON PERFECT SQUARE NUMBERS

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ABSTRACT

In this paper we discus the alternative approach to prove the irrationality of any number by taking advantages of continued fractions and significance of vector Euclidean algorithm by showing the special case of √2 and others. Also, we discuss irrationality of 2 by various proofs.

Keywords: Vector Euclidean algorithm, Continued fractions, Bezout lemma, rational root test etc.

INTRODUCTION

I saw the proof on different books for the $\sqrt{2}$ being irrational. I can follow all the steps, but the proof doesn't seem valid to me (*) [1] & [2]. I looked around the web and saw very similar descriptions for this proof and they too all seem invalid to me. All the proofs start with the idea that a rational number can be written as the ratio of two integers, say $\frac{a}{b}$, and that for any ratio there exists exactly one fully reduced fraction (where no integer greater than 1 exists that can be evenly divided into both the numerator and denominator.) What I see is there may exist a fraction that is not fully reduced. Even if I assumed a non-fully reduced fraction did exist, this does not imply to me there does not exist a non-fully reduced fraction. I would want to see that no fully reduced fraction of integers exists anywhere to see the conclusion.

(*) Let us assume that $\sqrt{2}$ is rational of the form $\frac{a}{b}$, where $\frac{a}{b}$ can be reduce to lowest terms. i.e., $\left(\frac{a}{b}\right)^2 = 2$, or $a^2 = 2b^2$. As we know that, a^2 is even, a must be even, say a = 2c $\Rightarrow (2c)^2 = 2b^2$, or $4c^2 = 2b^2 or 2c^2 = b^2$ $\Rightarrow$ b must also be even $\Rightarrow$ In $\frac{a}{b}$, both a and b are even, but we assumed we'd reduced the fraction to lowest terms. We've got an absurdity, $\Rightarrow \sqrt{2}$ should be irrational.

We are including a better idea to prove irrationality of 2 in this introduction part, as well as we generalize the same in the next section.

If we consider $\sqrt{2}$ is rational, then there exist a positive integer such that $\sqrt{2} \times q$ is again an integer and consider it is small. As we know that $1 < \sqrt{2} < 2 \Rightarrow \sqrt{2} - 1 < 1 \Rightarrow q(\sqrt{2} - 1) < q$, and consider this number as r. since $\sqrt{2}$ r is an integer $\Rightarrow$ $\sqrt{2}$ r = (q$\sqrt{2}$ - q) × $\sqrt{2}$ = (2q – q × $\sqrt{2}$) or in brief, *r* is a positive integer less than *q* and r × $\sqrt{2}$ is an integer. We have considered that q is the smallest positive integer and so we got an absurdity.

Generalization: Let us take [$\sqrt{n}$] is the integer part of $\sqrt{n}$. Here n is a non perfect square number (n = 2 etc). For any *n* that is not a perfect square, we may prove that $\sqrt{n}$ is irrational as we discussed above by considering $q \times (\sqrt{n} - [\sqrt{n}])$. Similarly, if *n* is a perfect square we get $\sqrt{n} = [\sqrt{n}]$, then there is no absurdity. Now the question arrives that, if *x* is a rational but not integral zero of a monic integer polynomial of degree *d*, let *q* be the least positive integer so that qx^j is an integral for every $j < d$ and letting $q(x - n)$ with *n* is an integer and $n < x < n + 1$, we get an absurdity. In other words, we have proved that all algebraic integers are integers.

Remark: The first such number was the golden ratio, (√5 +1)/2, as expressed in the ratio of lengths in a regular pentacle. I'm pretty sure this was the one that was first discovered by the Pythagoreans, allegedly the unfortunate Hypasus, subsequently denavisated and so on.

CONTINUED FRACTIONS

In mathematics, an infinite continued fraction is an infinite expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of *its* integer part and another reciprocal, and so on. A *finite continued fraction* is similar, but the iteration/recursion is terminated after finitely many steps by using an integer in lieu of another continued fraction. In either case, all integers in the sequence, other than the first, must be positive [3].

Here we will take $\sqrt{2}$ + 1 in the place of $\sqrt{2}$. Let us take a quadratic equation $m^2 - 2m - 1 = 0$, where one of the $m = \sqrt{2} + 1 \Rightarrow m = 2 + \frac{1}{v}$

$$\Rightarrow \sqrt{2} + 1 = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}}$$

This leads the above continued fraction. As we know that any number with infinitely repeats in simple continued fraction form is simply an *irrational*.

Now, we are introducing a direct proof to say the irrationality of 2 (and others) by taking an advantage of *Bezout* lemma [4].

Theorem 1: $k = \sqrt{n}$ *is integer rational for all* $n \in N$.

Proof: Let us take $k = \frac{a}{b}$ with (a, b) = 1 $\Rightarrow$ ad – bc = 1 for any c & d $\in$ Z.

By Bezout lemma, (a – bk) (c + dk) = 0 = ac – bdn + k $\Rightarrow$ k for k $\in$ Z.

Alternatively, we can prove the same for all non perfect square numbers in the following way.

Theorem 2: *The square roots of all non-perfect integers are irrational.*

Proof: Let us take x = √2 is an irrational by close look at (x + 1) and (x – 1). Here, (x – 1) = 1/(x + 1) and x – 1 = a/b gives us (a/b) + 2 = b/a, such that (a + 2b, b) = 1, which implies that a + 2b = b and b = a ⇒ a = 0 = b, which is an absurdity.[5]

Similarly, we can use the same idea, although we won't consider reciprocals. In fact, if √n is rational and then for any a/b with (a, b) = 1, we have $\sqrt{n} - \frac{a}{b} = \frac{c}{d}$ is a rational for some c and d are in Z. Of course, (c, d) =1 as d > 0. Now we see, $\frac{c}{d} + 2\frac{a}{b} = n - \frac{a^2}{b^2} \Rightarrow cb^2 = d(nb^2 - a^2 - 2ab) \Rightarrow b^2 | d(nb^2 - a^2 - 2ab)$. Since b^2 is co-prime to $(nb^2 - a^2 - 2ab)$, any prime factor of b^2 should be a prime factor of b and hence also of $nb^2 - 2ab$. If it were also of $(nb^2 - a^2 - 2ab)$, then it would divide a^2 and hence a, contrary to (a, b) = 1. Therefore, we have $b^2|d$. ⇒ $d = b^2r$.Then $c = r(nb^2-a^2-2ab)$. Now any prime factor of r divides c. Since r|d and(c, d)=1, r is a unit. As we have d > 0 and $b^2 > 0$, r > 0. So r = 1 ⇒ $d = b^2$ ⇒ $\sqrt{n} = \frac{a}{b} + \frac{c}{b^2} = \frac{(ab+c)}{b^2}$. By letting $\frac{a}{b} = 0$ and √n = c, an integer such that $n = c^2$ is a square of an integer. Thus, if n is not a perfect square integer, and then √n is an irrational.

Remark: We can also use the rational root test [6] for the polynomial equation $x^2 = 2$, which satisfies for x = ± 2. If this equation has a rational solution of the form a/b ⇒ a|2 as well as b|1 and then a/b ∈ {-2, -1, 1, 2}. However, none of this set of solutions is satisfying the above polynomial. Therefore $x^2 - 2 = 0$ has no rational roots and 2 is an irrational.

DISCUSSION BY UFD

We can alternatively prove the theorem 1 by the following way.

Theorem 3: *If a and b are positive integers, then $a^{1/b}$ is either irrational or integer.*

Proof: If $a^{1/b} = \frac{x}{y}$ and y does not divide x ⇒ $a = (a^{1/b})^b = x^b / y^b \notin Z$, as y^b does not divide x^b, which is an absurdity. We found a variant of this proof on [7] under UFD.
Lemma: If y does not divides x, then y^b does not divides x^b. UFD implies that there exists a prime p and positive integer t such that p^t divides y while p^t does not divide x. It implies that p^t divides y^b while p^{bt} does not divide x^t. Hence y^b does not divide x^b.

ANALYTICAL APPROACH

Lemma1: Let α ∈ R^+ and $p_1, p_2, \ldots, q_1, q_2, \ldots$ ∈ N such that $|\alpha q_n - p_n| \neq 0$ for all n ε N and $\lim_{n\to\infty}(p_n) = \lim_{n\to\infty}(q_n) = \infty$, $\lim_{n\to\infty}|\alpha q_n - p_n| = 0$ then α is irrational.

Proof: Let $\alpha = \frac{a}{b}$ with a,b ε N^+. For sufficiently large n, $0 < |\alpha q_n - p_n| < \frac{1}{b}$

$$\Rightarrow 0 < \left| aq_n / (b - p_n) \right| < \frac{1}{b}$$

$$0 < |aq_n - bp_n| < 1$$

But, $0 < |aq_n - bp_n| < 1$.
$\therefore$ α is irrational.

Theorem 4: $\sqrt{2}$ is irrational

Proof: Let $p_1 = q_1 = 1$ and $p_{n+1} = p_n{}^2 + 2q_n{}^2$

$q_{n+1} = 2p_n q_n$ for all n $\in$ N (which can be proved by induction)

$0 < \left|\sqrt{2}q_n - p_n\right| < 1/2^{2n-1}$ for all n $\in$ N. For n = 1

$0 < \left|\sqrt{2}q_1 - p_1\right| < 1/2$ As by induction, if it is true for n then it is true for n+1.

$0 < \left|\sqrt{2}(2p_n q_n) - (p_n{}^2 + 2q_n{}^2)\right| < 1/2^{2n}$,

$0 < \left|\sqrt{2}q_{n+1} - p_{n+1}\right| < 1/2^{2n}$,

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