American J. of Mathematics and Sciences
Vol. 3, No -1 ,(January 2014)
Copyright © Mind Reader Publications
ISSN No: 2250-3102

On the Structure of Some Groups Containing $M_9 wr M_{10}$

Basmah H Shafee
Department of Mathematics, Umm Al qura University,
Makkah, KSA.
e-mail: dr.basmah_1391@hotmail.com

Abstract

In this paper, we will generate the wreath product $M_9 wr M_{10}$ using only two permutations. Also, we will show the structure of some groups containing the wreath product $M_9 wr M_{10}$. The structure of the groups founded is determined in terms of wreath product $(M_9 wr M_{10})\, wr\, C_k$. Some related cases are also included. Also, we will show that S_{90K+1} and A_{90K+1} can be generated using the wreath product $(M_9 wr M_{10})\, wr\, C_k$ and a transposition in S_{90K+1} and an element of order 3 in A_{90K+1}. We will also show that S_{90K+1} and A_{90K+1} can be generated using the wreath product $M_9 wr M_{10}$ and an element of order $k+1$.

1. INTRODUCTION

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1}. They also gave the symmetric generating set of Groups A_{kn+1} and S_{kn+1} using S_n [5] .

Shafee [2] showed that the groups A_{kn+1} and S_{kn+1} can be generated using the wreath product A_m wr S_a and an element of order *k*+1. Also she showed how to generate S_{kn+1} and A_{kn+1} symmetrically using *n* elements each of order *k*+1.

Al-Amri and Eassa [6] have shown that the structure of some groups of degree9k containing M_9. They also shown that the the wreath product of the mathieu group M_{10} by some other groups [7] .

Al-Amri and Al-Shehri [3] have shown that S_{9k+1} and A_{9k+1} can be generated using the wreath product $M_9 \mathrm{wr} C_k$ and an element of order 4 in S_{9k+1} and element of order 5 in A_{9k+1}.

The Mathieu groups M_9 and M_{10} are two groups of the well known simple groups. In [8], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6,7] as follows :

$$M_9 =< X,Y \mid X^4 = Y^4 = [X,Y]^2 = (YXYX^3) = 1, [X,^2 XY] = (XY^{-2}X) >$$

$$M_{10} =< X,Y \mid X^5 = Y^4 = [X,Y]^3 = (XYXYX)^5 = (XY^2)^2 = 1 >$$

M_9 can be generated using two permutations, each of order 4 and an involution as follows : $M_9 =< (1,2,3,4)(5,6,7,8), (1,2,5,9)(3,6,8,7) >$.

M_{10} can be generated using two permutations, the first is of order 5 and the second of order 4 as follow:

$$M_{10} =< (1,2,3,4,5)(6,7,8,9,10), (1,7,4,9)(2,10,3,6) > .$$

In this paper, we will generate the wreath product $M_9 wrM_{10}$ using only two permutations. Also, we show the structure of some groups containing the wreath product $M_9 wrM_{10}$. The structure of the groups founded is determined in terms of wreath product $(M_9 wrM_{10})\, wr\, C_k$. Some related cases are also included. Also, we will show that S_{90K+1} and A_{90K+1} can be generated using the wreath product $(M_9 wrM_{10})\, wr\, C_k$ and a transposition in S_{90K+1} and an element of order 3 in A_{90K+1}. We will also show that S_{90K+1} and A_{90K+1} can be generated using the wreath product $M_9 wrM_{10}$ and an element of order $k+1$.

Keywords and phrases: wreath product, Mathieu group.

2. PRELIMINARY RESULTS

DEFINITION 2.1. Let *A* and *B* be groups of permutations on non empty sets Ω_1 and Ω_2 respectively. The wreath product of *A* and *B* is denote by *A* wr *B* and defined as *A* wr *B*= $A^{\Omega_2} \times_\theta B$, i.e., the direct product of | Ω_2 | copies of *A* and a mapping □ where □□ : *B* → Aut (A^{Ω_2}) is defined by $□_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that | *A* wr *B* | = $(|A|)^{□□□_2□}$ | *B* |.

THEOREM 2.2 [4] Let G be the group generated by the *n*-cycle (1, 2, …, *n*) and the 2-cycle (*n*, *a*). If $1 < a < n$ is an integer with *n* = *am* , then $G \cong S_m \text{ wr } C_a$.

THEOREM 2.3 [4] Let $1 \le a \ne b < n$ be any integers. Let *n* be an odd integer and let G be the group generated by the *n*-cycle (1,2,…,*n*) and the 3-cycle (n,a,b). If the $hcf(n,a,b)$=1, then $G = A_n$. While if *n* can be an even then $G = S_n$.

THEOREM 2.4 [4] Let $1 \le a < n$ be any integer. Let $G = \langle (1,2,...,n), (n,a) \rangle$. If *h.c.f.* $(n,a) = 1$, then $G = S_n$.

THEOREM 2.5 [4] Let $1 \le a \ne b < n$ be any integers. Let n be an even integer and let G be the group generated by the (*n*-1)-cycle $(1,2,...,n-1)$ and 3-cycle (*n*,*a*,*b*). Then $G = A_n$.

3. THE RESULTS

THEOREM 3.1 The wreath product $M_9 wrM_{10}$ can be generated using two permutations, the first is of order 90and the second is of order 4.

Proof : Let *G* = ⟨*X*,*Y*⟩, where: *X*=(1, 2, 3, 4, …, 90), which is a cycle of order 252, *Y*=(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27) (29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74), which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = \left((XY)^6 [X, Y]^5\right)^{18}$. Then

$$\alpha_1 = (10, 20, 30, 40, 50, 60, 70, 80, 90),$$

which is a cycle of order 9. Let $\alpha_2 = \alpha_1^{-1} X$. It is easy to show that

$$\alpha_2 = (1, 2, 3, \ldots, 10)(11, 12, 13, \ldots, 20) \ldots (81, 82, 83, \ldots, 90),$$

which is the product of seven cycles each of order 10. Let: $\beta_1 = (Y^2)^{(XY)^{18}}$ =(9, 20)(12, 23)(31, 53)(34, 56), $\beta_2 = \beta_1 Y^{-1}$ =(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74), $\beta_3 = (Y^3 \beta_2)^2$ =(1, 45)(12, 23), $\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$ =(10, 40)(50, 60) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (10, 60)(40, 50). Let $\alpha_3 = \beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$. Hence

$$\alpha_3 = (10, 20)(30, 50).$$

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

$$\alpha_4 = (1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28)(26,27)(29,30)(34,42)(35,39)(37,38)(40,41)(45,53)(46,50)(48,49)(51,52)(56,64)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74),$$

which is the product of twenty eight transpositions. Let $K = \langle \alpha_2, \alpha_4 \rangle$. Let $\theta : K \to M_{10}$ be the mapping defined by

$$\theta(10i+j) = j \quad \forall\ 1 \le i \le 8,\ \forall\ 1 \le j \le 10$$

Since $\theta(\alpha_2)$ =(1, 2, …, 10) and $\theta(\alpha_4)$ =(1, 9)(2, 6)(4, 5)(7, 8), then $K \cong \theta(K) = M_{10}$. Let $H_0 = \langle \alpha_1, \alpha_3 \rangle$. Then $H_0 \cong M_9)$. Moreover, *K* conjugates H_0 into H_1, H_1 into H_2 and so it conjugates H_{16} into H_0, where

$$H_i = < (i, 10+i, 20+i, 30+i, 40+i, 50+i, 60+i, 70+i, 80+i)(i, 10+i)(20+i, 40+i) >$$

$\forall\ 1 \le i \le 10$. Hence we get $M_9 wr M_{10}) \subseteq G$. On the other hand, Since *X*= $\alpha_1\alpha_2$ and *Y*= $\alpha_4\alpha_3^X$, then $G \subseteq M_9 wr M_{10}$. Hence $G = M_9 wr M_{10}$ ◊

THEOREM 3.2 The wreath product $(M_9 wr M_{10})\, wr\, C_k$ can be generated using two permutations, the first is of order 90*k* and an involution, for all integers *k* ≥ 1.

Proof : Let σ = (1, 2, …, 90*k*) and τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*) (59*k*, 60*k*) (62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*). If *k*=1, then we get the group $M_9 wr M_{10}$ which can be considered as the trivial wreath product $(M_9 wr M_{10})\, wr\, C_k$ wr<id>. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{10} \tau^{\sigma^{ik}}$, we get an element $\delta = \alpha^{45}$ =(*k*, 2*k*, 3*k*, …, 90*k*). Let $G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i} \rangle$, be the groups acts on the sets Γ_i ={ i, *k*+i, 2*k*+i,…, 89*k*+*i* }, for all $1 \le i \le k$. Since $\bigcap_{i=1}^{k} \Gamma_i = \varphi$, then we get the direct product $G_1 \times G_2 \times \ldots \times G_k$, where, by theorem 3.1 each $G_i \cong M_9 wr M_{10}$. Let $\beta = \delta^{-1}\sigma$ □□(1, 2, …, *k*)(*k*+1, *k*+2, …, 2*k*) … (89*k*+1, 89*k*+2, …, 90*k*). Let $H = \langle \beta \rangle \cong C_k$. H conjugates G_1 into G_2, G_2 into G_3,…and G_k into G_1. Hence we get the wreath product $(M_9 wr M_{10}) wr C_K \subseteq G$. On the other hand, since $\delta\beta$= (1, 2, …, *k*, *k*+1, *k*+2, …, 2*k*, …, 89*k*+1, 89*k*+2, …, 90*k*)=σ, then $\sigma \in (M_9 wrM_{10})wrC_K$. Hence $G = <\sigma, \tau> \cong (M_9 wrM_{10})wrC_K$.◊

THEOREM 3.3 The wreath product $(M_9\ wrM_{10})\,wr\,S_k$ can be generated using three permutations, the first is of order 90*k*, the second and the third are involutions, for all $k \geq 2$.

Proof : Let σ = (1, 2, …, 90*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*) and μ = (1, 2)(*k*+1, *k*+2)(2*k*+1, 2*k*+2) … (89*k*+1, 89*k*+2). Since by Theorem 3.2, $<\sigma, \tau> = (M_9\ wr\,M_{10})\,wr\,C_k$ and (1, 2, …, *k*)(*k*+1, *k*+2, …, 2*k*) … (89*k*+1, …,90*k*)∈ $(M_9\ wr\,M_{10})\,wr\,C_k$ then ⟨(1,…,*k*)(*k*+1,…,2*k*)…(89*k*+1,…,90*k*)□□ μ⟩ $\cong S_k$. Hence $G = \langle \sigma,\ \tau,\ \mu \rangle \cong (M_9 wrM_{10})wrS_k$. ◊

COROLLARY 3.4 The wreath product $(M_9\ wrM_{10})\,wr\,A_k$ can be generated using three permutations, the first is of order 90*k*, the second is an involution and the third is of order 3, for all odd integers $k \geq 3$.

THEOREM 3.5 The wreath product $(M_9 wr\,M_{10})\,wr\,(S_m\ \ wr\ C_a)$ can be generated using three permutations, the first is of order 90*k*, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

.

Proof : Let σ = (1, 2, …, 90*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*) and μ□= (*k*, *a*)(2*k*, *k*+*a*)(3*k*, 2*k*+*a*) … (90*k*, 891*k*+*a*). Since by Theorem 3.2, $<\sigma, \tau> \cong (M_9\ wrM_{10})\,wr\,C_k$ and (1, …, *k*)(*k*+1, …, 2*k*) … (89*k*+1, … ,90*k*)∈ $(M_9\ wrM_{10})\,wr\,C_k$ then

⟨(1, …, *k*)(*k*+1, …, 2*k*) …(89*k*+1, …, 90*k*□□ μ⟩≅ $(S_m \,\mathrm{wr}\, C_a)$. Hence $G = \langle \sigma,\ \tau,\ \mu \rangle \cong (M_9 wrM_{10})wr(S_m wrC_a)$.◊

THEOREM 3.6 S_{90k+1} and A_{90k+1} can be generated using the wreath product $(M_9\ wrM_{10})\,wr\,C_k$ and a transposition in S_{132k+1} for all integers $k > 1$ and an element of order 3 in A_{90k+1} for all odd integers $k > 1$.

Proof: Let σ = (1, 2, …, 90*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*)□□ μ =(90*k*+1,1) and μ' =(1,*k*, 902*k*+1) be four permutations, of order 90*k*, 2, 2 and 3 respectively. Let $H = \langle \sigma,\ \tau \rangle$. By theorem 3.2 $H \cong (M_9\ wrM_{10})\,wr\,C_k$.

Case 1: Let $G=\langle\sigma,\ \tau,\ \mu\rangle$. Let $\alpha=\sigma\mu$, then $\alpha=(1,2,\ldots,90k,90k+1)$ which is a cycle of order $90k+1$. By theorem 2.4 $G<\sigma,\tau,\mu'>\cong<\alpha,\mu>\cong\ S_{90k+1}$.

Case 2: Let $G=\langle\sigma,\ \tau,\mu'\rangle$. By theorem 2.5 $<\sigma,\mu'>\cong A_{90k+1}$. Since τ is an even permutation, then $G\cong A_{90k+1}$.

THEOREM 3.7 S_{90k+1} and A_{90k+1} can be generated using the wreath product $M_9\ wr\,M_{10}$ and an element of order $k+1$ in S_{90k+1} and A_{90k+1} for all integers $k\geq 1$.

Proof: Let $G=\langle\sigma,\ \tau\ ,\ \mu\rangle$, where, $\sigma=$(1, 2, 3, …, 90)(90(*k*-(*k*-1))+1, …, 90(*k*-(*k*-1))+90) … (90(*k*-1)+1, …, 90(*k*-1)+132), τ =(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) … (90(*k*-1)+1, 90(*k*-1)+9) … (90(*k*-1)+73, 90(*k*-1)+74), and μ =(90, 154, …,90*k*, 90*k*+1), where *k*-*i* >0, be three permutations of order 90, 4 and *k*+1 respectively. Let $H=\langle\sigma,\ \tau\rangle$. Define the mapping θ as follows ;

$$\theta(10(k-i)+j)=j\quad\forall\ 1\leq i\leq k\ ,\ \forall\ \ 1\leq j\leq 10$$

Hence $H=<\sigma,\tau>\cong M_9\ wr\,M_{10}$. Let $\alpha=\mu\sigma$ it is easy to show that $\alpha=(1,2,3,\ldots,90k+1)$, which is a cycle of order $90k+1$. Let $\mu'=\mu^{\sigma}=(1,91,...,90(k-1)+1,\,90k+1)$ and $\beta=[\mu,\ \mu']=(1,90,90k+1)$. Since $h.c.f\,(1,90,90k+1)$, then by theorem 2.3 $G=\langle\sigma,\tau,\mu\rangle\cong\langle\alpha,\beta\rangle\cong\ S_{90k+1}$ or A_{90k+1} depending on whether *k* is an odd or an even integer respectively. ◊

References :

[1] A. M. Hammas and I. R. Al-Amri, Symmetric generating set of the alternating groups A_{2n+1}, JKAU: Educ. Sci., 7 (1994), 3-7.

[2] B. A. Shafee, Symmetric generating set of the groups A_{kn+1} and S_{kn+1} using th the wreath product A_m wr S_a , Far East Journal of Math. Sci. (FJMS), 28(3) (2008), 707-711.

[3] H. A. AL-Shehri and I. A. Al-Amri, Symmetric and permutational generating sets of S_{9k+1} and A_{9k+1} using the wreath product M_9wrC_k, Far East J. Mathe. Sci. (FJMS), 31(2) (2008), 227-235.

[4] I.R. Al-Amri, Computational methods in permutation groups , ph.D Thesis, University of St. Andrews, September 1992.

[5] I.R. Al-Amri, and A.M. Hammas, Symmetric generating set of Groups A_{kn+1} and S_{kn+1}, JKAU: Sci., 7 (1995), 111-115.

[6] I.R. Al-Amri, and A.A. Eassa, On the Structure of some groups of degree9k containing M_9, Journal of Natural Sciences and Mathematics Qssim University, (2007), 82-85.

[7] I.R. Al-Amri, and A.A. Eassa, On the wreath product of the mathieu group M_{10} by some other groups, Journal of Taibah University for Science (JTUSCI), (2008), 72-75.

[8] J. H. Conway and others, Atlas of Finite Groups, Oxford Univ. Press, New York, 1985.