American Jr. of Mathematics and Sciences
Vol. 1, No. 1, January 2012

An "exposé" on Divisibility Test

K. Raja Rama Gandhi[1]
Resource person in Math, Oxford University Press-New Delhi & Department of Mathematics,
Chaitanya Engineering College, Kommadi, Madhurawada, Visakhapatnam -41.
Email: rrmath28@rediffmail.com

D. Ravi kumar[2]
Asst.Prof in Mathematics,
M V G R College of Engineering, Vizianagaram- 535005.
davalaravikumar@gmail.com

ABSTRACT

As we know that, every number has its own divisibility test/rule. In this paper, we will discuss the divisibility patterns of every number by the single formula and at the same time we will show the efficiency of the new method by introducing the theorem and by taking the advantage of congruencies.

INTRODUCTION

We are all familiar with divisibility tests for certain divisors such as *3*, *9* and *11*. In these tests, one can find the weighted sum of the digits of the dividend i.e., the digits are multiplied by a fixed set of weights which depend on the divisor.

$n = a_d 10^d + a_{d-1} 10^{d-1} + \ldots + a_1 10 + a_0,$ **(1)**
Where $0 \le a_i \le 9,$ we have;

$$n \equiv 0 \ (mod\ 3) \Leftrightarrow \quad a_d + a_{d-1} + a_{d-2} + \ldots + a_0 \ \ 0 \ (mod\ 3)$$
$$n \equiv 0 \ (mod\ 9) \ \Leftrightarrow a_d + a_{d-1} + a_{d-2} + \ldots + a_0 \ \ 0 \ (mod\ 9)$$
$$n \equiv 0 \ (mod\ 11) \Leftrightarrow a_d - a_{d-1} + a_{d-2} - \ldots + a_0 \ \ 0 \ (mod\ 11)$$

The above tests are painless to use because the sum of the digits of n and the alternating sum of the digits of n are much smaller than n, so we can turn the divisibility problem for n into a divisibility problem for a smaller number. Likewise, we can iterate the test again and again until we are left with a very small number to test.

A question naturally arises is "How does one can find the sequence of weights for the various divisors?" Answer to this is quite simple; however we shall see that the standard "right-to-left" method can be substantially improved for certain divisors. In that the sequence of weights given by the method just described is not always the best choice.

The test of *3*, *9* and *11*, generalize to a test for divisibility by any number m relatively prime to *10*. i.e. m is not multiple of *2* or *5*. For an example, divisibility test by *7*, *13* and *29*. The general test will involve the operating taking off the unit's digit of a positive integer. For an in stance, turning *1634 163*. For $n \ge 1$, let n' be the number that we get after taking all the unit digits of n. So if 'n' is written as in *(1)*.

$$n' = a_d 10^{d-1} + a_{d-1} 10^{d-2} + \ldots + a_1 \qquad \textbf{(2)}$$

Here we took off a_0 and shifted all the other digits in to the next lower position (a_1 fills the position previously taken by a_0, and so on…)

Theorem 1: If $(m, 10) = 1$, consider some b such that $10b \equiv 1 \ (mod\ m)$. Then,
$n \equiv 0 \ (mod\ m) \Leftrightarrow n' + ba_0 \equiv 0 \ (mod\ m)$.

Before writing a proof of the above cited theorem, let us observe, some examples before stating a theorem;

Example 1: Consider m = 7, then; 10 × 5 = 1(mod 7), therefore;
$n \equiv 0 \ (mod\ 7) \Leftrightarrow n' + 5a_0 \equiv 0 \ (mod\ 7).$ **(3)**

If we take n = 11382 and we have n' = 1138 and $n' + 5a_0 = 1138 + 5(2) = 1148 \Rightarrow 7|n$ if and only if 7|1148. Now, 1148 is little large number. So, we apply the test again to 1148. Now, 114 + 5(8) = 114 + 40 = 154, => 7|1148 if and only if 7|154. Again, 154 with 15 + 5(4) = 15 + 20 = 35, which is divisible by 7. i.e. the original number n = 11382 is divisible by 7. In other way, n = 11382 = 7(1626).

Let's summarize our successive computations in the following way:
11382 ⟿ 1138 + 5(2) = 1148 ⟿114 + 5(8) = 154 ⟿ 15 + 5(4) = 35.

If any b fitting 10b ≡ 1 (mod 7) can be used in place of 5 in this test. Since, 10(-2) ≡ 1 (mod 7), for instance, we also get a test for divisibility by 7 as

$n \equiv 0 \ (mod\ 7) \Leftrightarrow n' - 2a_0 \equiv 0 \ (mod\ 7)$ **(4)**

The above one is more convenient to use than (3) since -2 is smaller magnitude than 5.Of course (3) and (4) are the same test, as 5 ≡ -2 (mod 7). Let's apply (4) to 11382. The successive numbers we get now are;

11382 ⟿ 1138-2(2) = 1134 ⟿ 113-2(4) = 105 ⟿ 10-2(5) = 0, which is divisible by 7 so the 11382.

Let us discuss the proof of the theorem 1 below.

Proof:

Since $n = 10n^1 + a_0, n \equiv 0(\bmod m) \Leftrightarrow b(10n^1 + a_0)$

$$\equiv 0(\bmod m)$$

$$\Leftrightarrow n^1 + ba_0 \equiv 0(\bmod m)$$

All that really happened in the proof is that we divided by 10 working modulo m.
If we allow ourselves to use ordinary fractional notation, $10n^1 + a_0 \equiv 0(\bmod m)$

If and only if $n^1 + \frac{a_0}{10} \equiv 0(\bmod m)$ and the legal form of $\frac{1}{10}(\bmod m)\ is\ b(\bmod m)$ since $10b \equiv (\bmod m)$.
Although we said at the beginning that the divisibility test in theorem 1 generalize the divisibility test for 3,9,11, which involve adding (or alternatively adding and subtracting) all the digits of a number, the usual tests for 3, 9 and 11 don't actually look like the theorem 1. Let's see how the theorem 1 implies the usual tests for 3,9and11. by observations , we know that

$$n \equiv 0\ (\bmod 17) \Leftrightarrow n^1 - 5a_0 \equiv 0(\bmod 17)$$

$$n \equiv 0\ (\bmod 19) \Leftrightarrow n^1 + 2a_0 \equiv 0(\bmod 19)$$

$$n \equiv 0\ (\bmod 21) \Leftrightarrow n^1 - 2a_0 \equiv 0(\bmod 21)$$

$$n \equiv 0\ (\bmod 23) \Leftrightarrow n^1 + 7a_0 \equiv 0(\bmod 23)$$

$$n \equiv 0\ (\bmod 27) \Leftrightarrow n^1 - 8a_0 \equiv 0(\bmod 27)$$

$$n \equiv 0\ (\bmod 29) \Leftrightarrow n^1 + 3a_0 \equiv 0(\bmod 29)$$

m	b
3	1
7	-2
9	1
11	-1
$\vdots$	$\vdots$
27	-8
29	3
$\vdots$	$\vdots$

Table – 1

From the above table – 1,
B = 1 for m = 3 and 9, and b = -1 for m = 11
So, theorem 1 says us,

$$n \equiv 0 \pmod 3 \Leftrightarrow n^1 + a_0 \equiv 0 \pmod 3$$

$$n \equiv 0 \pmod 9 \Leftrightarrow n^1 + a_0 \equiv 0 \pmod 9$$

$$n \equiv 0 \pmod{11} \Leftrightarrow n^1 - a_0 \equiv 0 \pmod{11}$$

AS $10 \equiv 1 \pmod 3$ by (2)

$$n^1 \equiv a_d + a_{d-1} + \ldots + a_1 \pmod 3$$

$$n^1 + a_0 \equiv a_d + a_{d-1} + \ldots + a_1 + a_0 \pmod 3$$

Therefore the the test for divisibility by 3 in theorem 1 is the same as

$$n \equiv 0 \pmod 3 \Leftrightarrow a_d + a_{d-1} + \ldots + a_1 + a_0 \equiv 0 \pmod 3,$$

Which is the usual test for 3 . Since $10 \equiv 1 \pmod 9$,theorem 1 , implies the test for 9 in the same way. As for 11 , since $10 \equiv -1 \pmod{11}$, we have ,

$$\Leftrightarrow \quad (-1)^{d-1}\left(a_d - a_{d-1} + \ldots + (-1)^{d-1} a_1 + (-1)^d a_0\right) \equiv 0 \pmod{11}$$

$$\Leftrightarrow \quad a_d - a_{d-1} + \ldots + (-1)^{d-1} a_1 + (-1)^d a_0 \equiv 0 \pmod{11}$$

Which is usual for divisibility by 11 .

Theorem 2: the base 10 divisibility rule for 11. Sum the even numbered digits, subtract the odd- numbered digits, check (recursively) if result is divisible by 11 – can be generalized to a divisibility rule for n+1 in base n

Proof: let $x = \sum_{k=0}^{\infty} x_k n^k$ where n is the base and $x_{0,} x_{1,} x_2$ etc are the base-n digits of x.

since $n \equiv -1 \pmod{n+1}$, we have;

$n^k \equiv (-1)^k$ = 1 if k is even, -1 if k is odd (mod n+1)

And therefore $x = \sum_{k=0}^{\infty} x_k n^k \equiv \sum_{k=0}^{\infty} x_k (-1)^k$

$$= \sum_{k\,even}^{\infty} x_k - \sum_{k\,odd}^{\infty} x_k \pmod{n+1}$$

Thus x is divisible by n+1 if and only if the sum of its even base – n digits minus the sum of its odd base – n digits is divisible by n+1.

REFERENCES

[1] James Van Dyke, James Rogers, Hollis Adams, Fundamentals of Mathematics, 9th edition,
[2] Vivienne P. Joannou, Step-by-step maths