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# IMPLICIT NUMERICAL METHOD FOR SPACE FRACTIONAL ANOMALOUS DIFFUSION EQUATION

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**Abstract**

The aim of this paper is to develop the implicit finite difference scheme for space fractional diffusion equation with initial and boundary conditions. We also prove the scheme is unconditionally stable and convergent. Also, as an application of this scheme, the numerical solution for space fractional diffusion equation is obtained and represented graphically by Mathematica software.

**Keywords:** Implicit scheme, finite difference, space fractional, diffusion equation, stability analysis, convergence analysis, Mathematica.

# 1 INTRODUCTION

The fractional differential equations play a pivotal role in the modeling of number of physical phenomenon [1, 2]. The applications of such include damping laws, fluid mechanics, viscoelasticity, biology, physics, hydrology, engineering, finance, modeling of earth quakes etc.,[5, 12, 13]. One of the most important applications that has been rigorously and extensively studied is to describe the sub-diffusion and super-diffusion process [3, 6, 9, 11]. A mathematical approach to model such anomalous diffusion phenomena is based on generalized diffusion equation containing derivatives of fractional order in space or time or space-time. But due to the complexities of the physical nature of the problem, obtaining an exact solution becomes almost impossible. Also, most of the analytical methods used so far, encounter some inbuilt deficiencies and are not compatible with the true physical nature of these problems. Hence, approximation and numerical techniques must be used, as the numerical results reveal the complete reliability of the proposed algorithms [4, 14, 15, 18]. Based on these, our main purpose of this paper is to develop the space fractional implicit finite difference scheme for diffusion equation of fractional order [7, 8, 10, 17]. The model IBVP for space fractional heat transfer equation is given as,

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}},\ 0 < x < L,\ 1 < \beta \leq 2,\ t > 0$$

$$initial\ condition : U(x,0) = \phi(x),\ 0 \leq x \leq L$$

$$\textit{boundary conditions}: U(0,t) = 0 \textit{ and } U(L,t) = 0 \; 0 \leq t \leq T$$

where diffusion coefficient $D > 0$.
We consider the following definition of fractional derivative which is useful for our further developments.

**Definition 1.1** *The Grunwald-Letnikov space fractional derivative of order* $\beta$*,* $(1 < \beta \leq 2)$ *is defined as follows [10, 12, 13]*

$$\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} = \frac{1}{\Gamma(-\beta)} \lim_{N\to\infty} \frac{1}{h^{\beta}} \sum_{j=0}^{N} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} U(x-(j-1)h,t)$$

*where* $\Gamma(.)$ *is the gamma function.*

The paper is organised as follows: The implicit discrete approximation scheme for the space fractional diffusion equation is developed in sec 2. The unconditional stability of the solution is proved in section 3 and the concept of convergence of the scheme is discussed in section 4. We obtain the 3-D graphics of the approximate solution by the programming language Mathematica, followed by concluding remarks, in the last section.

## 2 SPACE FRACTIONAL FINITE DIFFERENCE SCHEME

We consider the space fractional diffusion equation with initial and boundary conditions as follows

$$\frac{\partial U(x,t)}{\partial t} = D \frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}}, t > 0, \; 0 \leq x \leq L, 1 < \beta \leq 2, \tag{2.1}$$

$$\textit{initial condition}: \; U(x,0) = \phi(x), \; 0 \leq x \leq L \tag{2.2}$$

$$\textit{boundary conditions}: U(0,t) = 0 \textit{ and } U(L,t) = 0 \; 0 \leq t \leq T \tag{2.3}$$

where D is the diffusivity constant. Now, for the implicit numerical approximation scheme, we first discretize the domain into a fine grid of equal rectangles of sides $\delta x = h$ and $\delta t = \tau$. That is, we define $\delta x = h = \frac{L}{N}$ and $\delta t = \tau = \frac{T}{N}$, the space and time steps respectively such that $t_k = k\tau$; $0 \leq t_k \leq T$; $k = 0, 1, 2...N$ be the integration time and $x_i = x_L + ih$ for $i = 0, 1, 2...N$. We define $U_i^k = U(x_i, t_k)$ and let $U_i^k$ denote the numerical approximation to the exact solution $U(x_i, t_k)$. In the partial differential equation (2.1), we discretize the spatial $\beta$-order fractional derivative using the *Grünwald* finite difference formula at all time levels. The standard *Grünwald* estimate generally yields unstable finite difference equation regardless of whatever result in finite difference method is an explicit or an implicit system for related discussion [10, 13]. Therefore, we use a right shifted *Grünwald* formula to estimate the spatial $\beta$-order fractional derivative.

$$\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} = \frac{1}{\Gamma(-\beta)} \lim_{N\to\infty} \frac{1}{h^{\beta}} \sum_{j=0}^{N} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} U(x-(j-1)h,t)$$

where N is the positive integer and $\Gamma(.)$ is the gamma function. We also define the normalized *Grünwald* weights by

$$g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)},\ j = 0, 1, ...$$

Using the right shifted *Grünwald* formula, the implicit type numerical approximation to equation (2.1), we get

$$\frac{U_i^{k+1} - U_i^k}{\tau} = D\ \delta_{\beta,x}\ U_i^{k+1}$$

where the above fractional partial differential operator is defined as

$$\delta_{\beta,x} U_i^k = \frac{1}{h^\beta} \sum_{j=0}^{i+1} g_{\beta,j} U_{i-j+1}^k$$

which is an $O(h^\beta)$ approximation to the $\beta$-order fractional derivative. Therefore, the fractional approximated equation is,

$$\frac{U_i^{k+1} - U_i^k}{\tau} = \frac{D}{h^\beta} \sum_{j=0}^{i+1} g_{\beta,j} U_{i-j+1}^{k+1}$$

By setting $\frac{D\tau}{h^\beta} = r$, the above equation is,

$$\Rightarrow (1 - r g_{\beta,1}) U_i^{k+1} - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} U_{i-j+1}^{k+1} = U_i^k,\ i = 1, ...N$$

For $k = 0$, we get

$$(1 - r g_{\beta,1}) U_i^1 - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} U_{i-j+1}^1 = U_i^0,\ i = 1, ...N.$$

The initial condition is approximated as $U_i^0 = \phi(ih); i = 1, 2...N$ and the boundary conditions are approximated as $U_0^k = 0$ and $U_L^k = 0$. Therefore, the fractional approximated IBVP is

$$(1 - r g_{\beta,1}) U_i^1 - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} U_{i-j+1}^1 = U_i^0,\ for\ k = 0 \tag{2.4}$$

$$(1 - r g_{\beta,1}) U_i^{k+1} - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} U_{i-j+1}^{k+1} = U_i^k,\ for\ k \geq 1 \tag{2.5}$$

$$initial\ condition:\ U_i^0 = \phi(ih);\ i = 1, 2...N \tag{2.6}$$

$$boundary\ conditions:\ U_0^k = 0\ and\ U_L^k = 0 \tag{2.7}$$

where $r = \frac{D\,\tau}{h^{\beta}}$.
Therefore, for $i = 1, 2, ...N-1$, from the above equations, the fractional approximated IBVP (2.4) to (2.7) can be written in the following matrix equation form

$$AU^{k+1} = U^k \tag{2.8}$$

where $U^k = (U_1^k, U_2^k, ..., U_{N-1}^k)^T$ and $A = (a_{ij})$ is a square matrix of coefficients of order N-1. For $i = 1, 2, ..., N-1,\ j = 1, 2, ...., N-1$ the coefficients are

$$a_{ij} = \begin{cases} 0,\ when\ j \geq i+2 \\ -rg_0,\ when\ j = i+1 \\ 1 - rg_1,\ \ when\ j = i = 1, 2, 3, ...., \\ -rg_j,\ \ otherwise\ j = 2, 3, 4, ...., N-1 \end{cases} \tag{2.9}$$

where $r = \frac{D\tau}{h^{\beta}},\ g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}$. The above system of algebraic equations is solved by using Mathematica software in the last section.
In the next section, we discuss the stability of the solution of space fractional implicit finite difference scheme $(2.4)-(2.7)$ for the space fractional diffusion equation $(2.1)-(2.3)$.

# 3 STABILITY

This section is devoted for the stability of the fractional implicit finite difference scheme $(2.4) - (2.7)$ for the space fractional diffusion equation $(2.1) - (2.3)$.
**Lemma 3.1:** If $\lambda_j(A),\ j = 1, 2, ..., N-1$ represent eigenvalues of matrix A then we prove the following results:
(i) $|\lambda_j(A)| > 1$, j = 1, 2, ... , N-1.
(ii) $\|A^{-1}\|_2 \leq 1$
Proof: The Gerschgorin theorem states that each eigenvalue $\lambda$ of a square matrix A is in at least one of the following disk

$$|\lambda - a_{jj}| \leq \sum_{l=1, l\neq j}^{N-1} a_{lj}, j = 1, 2, ..., N-1. \tag{3.1}$$

Therefore, each eigenvalue $\lambda$ of matrix A satisfies at least one of the following inequalities:

$$|\lambda| \leq |\lambda - a_{jj}| + \sum_{l=1, l\neq j}^{N-1} |a_{lj}| \leq \sum_{l=1}^{N-1} |a_{lj}| \tag{3.2}$$

$$|\lambda| \geq |a_{jj}| - |\lambda - a_{jj}| \geq |a_{jj}| - \sum_{l=1, l\neq j}^{N-1} |a_{lj}| \tag{3.3}$$

To prove (i), we use equation (3.3) to matrix A, then each eigenvalue $\lambda$ of matrix A satisfies the following inequality.

$$\begin{aligned}
|\lambda_1(A)| &\geq |1 - rg_1| - |-rg_0| \\
&\geq 1 - rg_1 - rg_0 \\
&\geq 1 - r(g_0 + g_1) > 1, \ (since \ r(g_0 + g_1) < 0) \\
|\lambda_1(A)| &> 1, \\
|\lambda_2(A)| &\geq |1 - rg_1| - |-rg_2 - rg_0| \\
&\geq 1 - rg_1 - (rg_2 + rg_0) \\
&\geq 1 - r(g_0 + g_1 + g_2) > 1, \ (since \ r(g_0 + g_1 + g_2) < 0) \\
&\ \cdot \ \cdot \ \cdot \\
|\lambda_{N-1}(A)| &\geq |1 - rg_1| - |-rg_{N-1} - rg_{N-2} ... - rg_2| \\
&\geq 1 - r(g_{N-1} + g_{N-2} + ... + g_2 + g_1) > 1 \\
|\lambda_{N-1}(A)| &> 1
\end{aligned}$$

Therefore, this proves $|\lambda_j(A)| > 1$, j = 1, 2, ... , N-1.
To prove (ii), we have $\|A\|_2 = \max\limits_{1 \leq j \leq N-1} |\lambda_j(A)| > 1$

$$\|A^{-1}\|_2 \leq \frac{1}{|\lambda_j(A)|} \leq 1$$

**Theorem 3.1** *The solution of the fractional approximated IBVP* (2.4) – (2.7) *is unconditionally stable.*

Proof: To prove that the above scheme is unconditionally stable.
We must show that $\|U^k\|_2 \leq \|U^0\|_2 \ for \ k \geq 1$.
From the equation (2.8), we have

$$AU^k = U^{k-1}, k = 1, 2, ... \tag{3.4}$$

Clearly, matrix A is invertible. Now for k = 1, 2, ..., from equation (3.4), we get

$$\begin{aligned}
AU^1 &= U^0 \\
U^1 &= A^{-1}U^0 \\
AU^2 &= U^1 \\
U^2 &= A^{-1}U^1 \\
&= A^{-1}(A^{-1}U^0) \\
U^2 &= (A^{-1})^2 U^0 \\
&\vdots \\
U^k &= (A^{-1})^k U^0, \ n \geq 1 \qquad (3.5)
\end{aligned}$$

From equation (3.5), we get

$$\|U^k\|_2 \leq \|A^{-1}\|_2^k \|U^0\|_2, \quad (By\ lemma\ 3.1,\ \ \|A^{-1}\|_2 \leq 1)$$
$$\|U^k\|_2 \leq \|U^0\|_2$$

This shows that the finite difference scheme for fractional equation is unconditionally stable.
Hence the proof is completed.

# 4 CONVERGENCE

We now discuss the convergence of the finite difference scheme. Consider the vector $\vec{U}^n = [U(x_0,t_n),...,U(x_i,t_n),...,U(x_N,t_n)]^T$ which represents the exact solution at time level $t_n$, whose size is N. The finite difference scheme (3.4) will become

$$A\vec{U}^n = \vec{U}^{n-1} + \tau^n, \ n = 1, 2, .... \tag{4.1}$$

where $\tau^n$ is the vector of the truncation errors at level $t_n$.

**Theorem 4.1** *The fractional order finite difference scheme* $(2.4) - (2.7)$ *for space fractional diffusion equation is convergent.*

Proof: If we subtract (3.4) from (4.1), we get

$$A(\vec{U}^n - U^n) = (\vec{U}^{n-1} - U^{n-1}) + \tau^n \tag{4.2}$$

Consider the error vector, $E^n = \vec{U}^n - U^n$, then from equation (4.2), we get

$$A\ E^n = E^{n-1} + \tau^n \tag{4.3}$$

In equation (4.3), putting n = 1, 2, ..., we get

$$\begin{aligned}
A\ E^1 &= E^0 + \tau^1 \\
E^1 &= A^{-1}\ E^0 + A^{-1}\tau^1 \\
A\ E^2 &= E^1 + \tau^2 \\
E^2 &= A^{-1}\ E^1 + A^{-1}\tau^2 \\
&= A^{-1}[A^{-1}\ E^0 + A^{-1}\tau^1] + A^{-1}\tau^2 \\
&= (A^{-1})^2\ E^0 + A^{-1}[A^{-1}\tau^1 + \tau^2] \\
&\ \vdots
\end{aligned}$$

$$E^n = (A^{-1})^n\ E^0 + A^{-1} \sum_{k=0}^{n-1} (A^{-1})^k\ \tau^{n-k} \tag{4.4}$$

We take $U^0 = \vec{U}^0, then\ E^0 = 0$ is a zero vector, then from (4.4), we get

$$\|E^n\|_2 \leq \|A^{-1}\|_2(\sum_{k=0}^{n-1} \|A^{-1}\|_2^k) \cdot \max_{1 \leq M \leq n} \|\tau^M\|_2 \tag{4.5}$$

Since by Lemma (3.1), $\|A^{-1}\|_2 \leq 1$ and $\lim_{(h,\tau)\to(0,0)} \|\tau^M\|_2 = 0, \quad (1 \leq M \leq n)$
Therefore, from equation (4.5), we get

$$\|E^n\|_2 \to 0,\ as\ (h,\tau) \to (0,0)$$

. Hence, the proof.

# 5 NUMERICAL SOLUTIONS

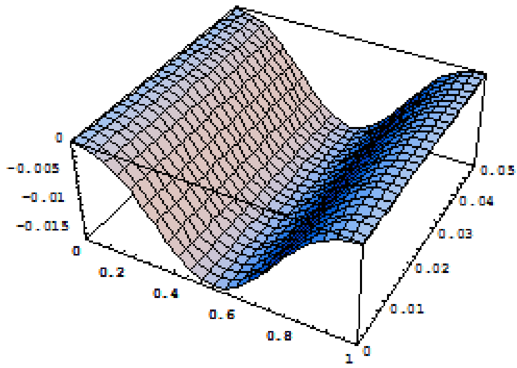
In this section, we obtain the approximated solution of space fractional diffusion equation with initial and boundary conditions. To obtain the numerical solution of the space fractional diffusion equation by the finite difference scheme, it is important to use some analytical model. Therefore, we present an example to demonstrate that the implicit finite difference scheme can be applied to simulate behavior of a fractional diffusion equation by using Mathematica software. We consider the following, one-dimensional space fractional diffusion equation with suitable initial and boundary boundary conditions

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^\beta U(x,t)}{\partial x^\beta}\ 0 < x < 1,\ 1 < \beta \leq 2,\ t > 0$$

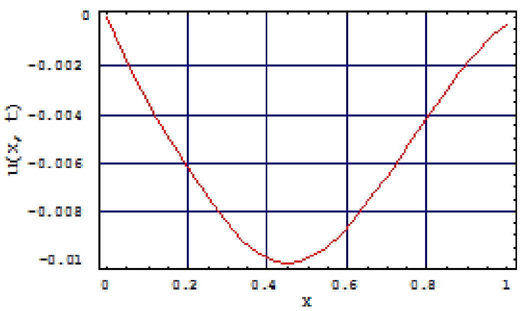
$$initial\ condition : U(x,0) = (x-1)^3 * x^3,\ \ 0 \leq x \leq 1$$

$$boundary\ conditions : U(0,t) = 0,\ \ and\ U(1,t) = 0$$

with the diffusion coefficient $D = 1$.
The numerical solution obtained at $t = 0.05$ by considering the parameters $\tau = 0.005$, $h = 0.1$, $\beta = 1.7$ is simulated in the following figure.




$Fig.5.1 : The\, exact\, diffusion\, profile\, withparameters\ t = 0.05,\ h = 0.1,\ \beta = 1.7$




$Fig.5.2 : The\, numerical\, diffusion\, profile\, with\, parameters\ t = 0.05,\ h = 0.1,\ \beta = 1.7$

**CONCLUSIONS**

(i) We developed the fractional order finite difference scheme for space fractional diffusion equation.

(ii) The numerical example is analyzed to show that the numerical results are in good agreement with theoretical analysis.

(iii) The fractional order implicit finite difference scheme is numerically stable.

# References


[1] **K.T. Alligood, T.D. Sauer, J.A. Yorke**, *An Introduction to Dynamical Systems*, Springer (1996).

[2] **R. L. Bagley and P. J. Torvik**, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech.,51, 294-298 (1984).

[3] **A. V. Chechkin, R. Goreno, I. M. Sokolov**, *Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations*, Phys. Rev. E. 66 (2002).

[4] **K. Diethelm**, *An algorithm for the numerical solution of differential equations of fractional order*, Electron. Trans. Numer. Anal.,5, 1-6 (1997).

[5] **M.M. Khader**, *On the numerical solutions for the fractional diffusion equation, Communications in Nonlinear Science and Numerical Simulation*, 16,2535-2542 (2011).

[6] **N. Krepysheva, L. D. Pietro, M. C. Neel**, *Space-fractional advection-diffusion and reflective boundary condition*, Phys. Rev. E. 73 (2006).

[7] **F.Liu, P. Zhuang, V. Anh, I, Turner**, *A Fractional Order Implicit Difference Approximation for the Space-Time Fractional Diffusion equation ,ANZIAM J.47 (EMAC2005)*, pp.C48-C68:(2006)

[8] **C. Lubich**, *Discretized Fractional Calculus*, SIAM J. Math. Anal., 17,704-719(1986).

[9] **R. Metzler, J. Klafter**, *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. 339, 1-77 (2000).

[10] **K. Miller, B. Ross**, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Eiley, New York ( 1993).

[11] **D. C. Negrete, B. A. Carreras, V. E. Lynch**, *Front Dynamics in Reaction-Diffusion Systems with Levy Flights: A Fractional Diffusion Approach*, Phys. Rev. Lett. 91 (2003).

[12] **K. B. Oldham and J. Spanier**, *The Fractional Calculus. Acad. Pre. New York and London*, (1974).

[13] **I. Podlubny**, *Fractional Differential equations*, Academic Press, San Diago (1999).

[14] **Qianqian Yang, Ian Turner, Fawang Liu**, *Analytical and Numerical Solutions for the time and Space-Symmetric Fractional Diffusion Equation*, ANZIAM J. 50 (CTA 2008) pp. C800-C814, (2009).

[15] **A. Quarteroni, A. Valli**, *Numerical Approximation of Partial Differential Equations*, Springer (1997).

[16] **B.Ross (Ed)**, *Fractional Calculus and Its Applications, Lecture Notes in Mathematics*, Vol.457, Springer-Verlag, New York ( 1975).

[17] **S. Shen, F.Liu**, *Error Analysis of an explicit Finite Difference Approximation for the Space Fractional Diffusion equation with insulated ends*, *ANZIAM J.46(E)*, pp.C871-C887:(2005).

[18] **N. H. Sweilam, M. M. Khader and A. M. Nagy**, *Numerical solution of two-sided space-fractional wave equation using finite difference method*, Journal of Comput. and Applied Mathematics,235, 2832-2841,(2011).