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On Almost Locally Compact Spaces

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Abstract

The aim of this paper is to introduce and give preliminary investigation of almost locally compact spaces (ALC). Every Hausdorff locally compact group is ALC .The concepts are equivalent in the class of semi-regular spaces. Almost locally compact is hereditary for regular open subspaces. It is preserved by open onto almost continuous maps and is a topological property.

Key words: locally compact ; almost locally compact; regular open set; nearly-compact.
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Introduction

This paper introduces the concept of an almost locally compact space. Every Hausdorff locally compact space is almost locally compact, and the concepts are equivalent in the semi-regular spaces. Almost locally compact is hereditary for regular open subspaces. It is preserved by open onto almost continuous maps. In section 1 we recall some definitions to be used in the sequel . In section 2 we define the concept of almost locally compact space .

1 Preliminaries

A neighborhood of a point p in a space X will mean an open subset of X containing p. If $A \subseteq X$, the closure and interior of A is denoted by cl(A) , int(A), respectively.
An open set G in X is called *regular open* if $G = int(cl(G))$. A regular open neighborhood of a point p in X will mean a regular open set containing p. A space is said to be *semi-regular* if X has a basis consisting of regular open sets [11]. A map $f : X \to Y$ is *almost continuous if* $f^{-1}(U)$ *is open in* X *whenever* U *is regular open in* Y. *A map* $f : X \to Y$ *ia almost open provided* $f(U)$ *is open in* Y *whenever* U *is regular open in* X.*A topological space* X *is said to be zero-dimensional if its topology has a basis consisting of open closed sets.*
A topological space (X, τ) *is said to be nearly-compact if every cover of* X *by regular open sets has a finite subcover. Clearly every compact space is nearly-compact. Considering the Sinrinov topology [10,Example 64] on the closed unit interval of reals , we obtain an example which is nearly-compact but not locally compact. For more on nearly-compact see[1],[3],[7],[8],[9]. We make the following*

notations:
ALC:almost locally compact space ; LC: locally compact space.

2 Almost locally compact spaces

In this section we introduce the concept of an almost locally compact space .We will show that this property is preserved under an almost continuous map and is a topological property.

Definition 2.1. *A space X is almost locally compact (denoted by ALC) at $p \in X$ if given a regular open neighborhood U of p, there is a neighborhood W and a nearly-compact set C such that $a \in W \subseteq C \subseteq U$.*
The space X is ALC if X is ALC at each of its point.

Remark 2.2.

(a) Since X itself is a regular open set , it is clear that each point in an ALC space is contained in a nearly-compact set.

(b) Every Hasudorff locally compact space is ALC, but example 2.3 shows that the converse is not true.

Example 2.3 . Let R be the reals and τ the topology over R having the open intervals and set $A = \{x \in R; x \textit{ is rational and } 1/3 < x < 2/3\}$ as a subbasis. Note that A does not contain the closure of any of its open sets, since such closure must be identical to the usual Euclidean closure. So A is not a regular open set. Hence the regular open sets in $[0, 1]$ are intervals. Given the set $X = [0, 1]$ the subspace topology of (R, τ) , it follows that $X = [0, 1]$ is an ALC Hasudorff space which is not LC [3].

The two concepts , LC and ALC , are equivalent in the class of semi-regular spaces.
Lemma 2.4. *Let X be a Hausdorff semi-regular space. Then X is ALC if and only if X is LC.*

Proof. It is clear that LC implies ALC. Conversely, let X be an ALC space , $x \in X$. Since X is semi-regular so for every open set U of x there is a regular open set G such that $x \in G \subseteq U$. By definition there exists a nearly compact set K and a neighborhood V such that $x \in V \subseteq K \subseteq G \subseteq U$. It is easy to show that in a semi-regular space every nearly compact is compact. Hence , K is compact . Therefore, X is locally compact.

As an immediate result we have:

Corollary 2.5. *Let X be a zero -dimensional Hausdorff space. Then X is ALC if and only if X is LC.*

Proof. Let X be a zero -dimensional Hausdorff space. It is clear that X is a semi-regular space. By lemma 2.4, the result holds.

Theorem 2.6. *If $\{G_\alpha\}$ is a family of regular open subsets of $\{X_\alpha\}$ such that for each α, $G_\alpha \subseteq X_\alpha$ then $\prod G_\alpha$ is regular open in $\prod X_\alpha$.*

proof. Let $\{G_\alpha\}$ be a family of regular open subsets of $\{X_\alpha\}$ such that for each α, $G_\alpha \subseteq X_\alpha$. Suppose $(x_\alpha) \in \prod X_\alpha$ and $(x_\alpha) \in int(cl(\prod G_\alpha))$. Then there exists $\prod U_\alpha$ a neighborhood of (x_α) such that $(x_\alpha) \in \prod U_\alpha \subseteq (cl(\prod G_\alpha))$. So $\prod G_\alpha \bigcap \prod U_\alpha \neq \emptyset$ i.e. for each α, $G_\alpha \bigcap U_\alpha \neq \emptyset$ and $x_\alpha \in cl(G_\alpha)$. Hence , $(x_\alpha) \in \prod cl(G_\alpha)$. So $int(cl(\prod G_\alpha)) \subseteq int \prod cl(G_\alpha) \subseteq \prod int(cl(G_\alpha)) = \prod G_\alpha$, since G_α is regular open for each α. Therefore, $\prod G_\alpha$ is regular open.

Proposition 2.7. *The product of a finite number of ALC spaces is ALC.*

Proof. Let X , Y be ALC spaces and $(p, q) \in X \times Y$. Let G and H be regular open sets containing p, q , respectively. By definition there exist nearly-compact sets K ,L and open sets U ,V such that

$$p \in U \subseteq K \subseteq G$$

$$q \in V \subseteq L \subseteq H$$

It is known that the product of two nearly compact sets is nearly compact [3]. So, $K \times L$ is nearly-compact. By theorem 2.6, $G \times H$ is regular open. Hence,

$$(p, q) \in U \times V \subseteq K \times L \subseteq G \times H$$

So $X \times Y$ is ALC.

Lemma 2.8 . *Let X ,Y be topological spaces and $f : X \to Y$ is an open onto almost continuous map. If X is nearly-compact then so is Y.*

Proof . Let X ,Y be topological spaces and $f : X \to Y$ is an open onto almost continuous map. Suppose X is nearly-compact and $\{V_\alpha\}$ is an open cover of Y by regular open sets. By [4,lemma 17.3] each $f^{-1}(V_\alpha)$ is a regular open set . So $\{f^{-1}(V_\alpha)\}$ is an open cover of X by regular open sets. Since X is nearly compact , then $X = \cup_1^n f^{-1}(V_{\alpha_i})$. Hence $Y = \cup_1^n V_{\alpha_i}$. Therefore, Y is nearly-compact.

The following proposition shows that ALC is a topological property.

Proposition 2.9. *Let X be an ALC space , Y a topological space $f : X \to Y$ an open onto and almost continuous map. Then Y is ALC.*

Proof. Let X be an ALC space , Y a topological space $f : X \to Y$ an open onto and almost continuous map. Suppose $y \in Y$ and G is a regular open set containing y. Let $y = f(x)$ for some $x \in X$. By [4,Lemma 3.17], $f^{-1}(G)$ is a regular open set. Hence, there exists an open set U and a nearly-compact set K such that

$$x \in U \subseteq K \subseteq f^{-1}(G)$$

Now $y \in f(U) \subseteq f(K) \subseteq G$. Since f is an open map , then $f(U)$ is open. By lemma 2.8, $f(K)$ is nearly-compact. Hence , Y is ALC.

Corollary 2.10 . *ALC is a topological property.*
Proof. It is clear by proposition 2.9.

Recall that a function $f : X \to Y$ is called super-continuous (abbreviated as SC) if the inverse image of every open set in Y is regular open in X.

It is clear that every SC map is continuous but the converse may not hold [7].

Lemma 2.11. *Suppose $f : X \to Y$ is an open onto SC map. If X is an ALC , then Y is LC.*

Proof. Let X be an ALC space and $f : X \to Y$ an open onto SC map .Let $y \in Y$, U an open set in Y containing y. By definition $f^{-1}(U)$ is a regular open set containing x. Since X is ALC then there exist an open set W and a nearly-compact set C such that $x \in W \subseteq C \subseteq f(U)$. Hence, $y \in f(W) \subseteq f(C) \subseteq U$. Now $f(U)$ is open . By [5,proposition 13], $f(C)$ is compact. So Y is LC.

Remark. Let U be a regular open set in the space $\prod X_\alpha$ with the usual product topology. By noting the behavior of the closure and interior operators on basic open sets, it follows that there exists a finite collection of regular open sets $\{U_{\alpha_i}\}_{i=1}^n$ (U_{α_i} regular open in X_{α_i})and regular open set $G = \prod_{\alpha \neq \alpha_i} X_\alpha \times U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}$ containing (x_α) such that $(x_\alpha) \in G \subset U$.

We have the following product theorem for ALC spaces

Theorem 2.12. *Let $\{X_\alpha\}$ be a family of topological spaces. Then $\prod X_\alpha$ is ALC if and only if all the X_α are ALC and at most finitely many are not nearly compact.*

Proof. Let $\{X_\alpha\}$ be a family of topological spaces. Suppose all the X_α are ALC and at most finitely many are not nearly-compact. Let $(x_\alpha) \in \prod X_\alpha$ and U is a regular open set containing (x_α). By the remark there exists a finite collection of regular open sets $\{U_{\alpha_i}\}_{i=1}^n$ (U_{α_i} regular open in X_{α_i})and regular open set $G = \prod_{\alpha \neq \alpha_i} X_\alpha \times U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}$ containing (x_α) such that $(x_\alpha) \in G \subset U$. Since each X_α is ALC so there are an open set V_{α_i} and a nearly-compact set C_{α_i} such that $x_{\alpha_i} \in V_{\alpha_i} \subset C_{\alpha_i} \subset U_{\alpha_i}$ $i = 1.2....n$. It is Known that the product of nearly compact is nearly-compact[3]. Hence

$$C = \prod C_{\alpha_i} \times \prod_{\alpha \neq \alpha_i} X_\alpha$$

is nearly-compact. Hence,$(x_\alpha) \in (\prod V_{\alpha_i})_{i=1}^n \times \prod_{\alpha \neq \alpha_i} X_\alpha \subset C \subset G \subset U$. Therefore, $\prod X_\alpha$ is ALC.

Conversely, assume that $\prod X_\alpha$ is an ALC space. Since each projection map $P_\alpha : \prod X_\alpha \to X_\alpha$ is continuous open surjection, by lemma 2.9 each X_α is ALC. Since $\prod X_\alpha$ is ALC , for every $(x_\alpha) \in \prod X_\alpha$ there are an open set U and a nearly-compact set C such that $(x_\alpha) \in U \subset C \subset \prod X_\alpha$. Now by lemma 2.8 ,$P_\alpha(C)$ is nearly compact for each α. Since $P_\alpha(C) = X_\alpha$ for all but at most finitely many indices α , the result follows.

Unlike locally compact spaces, an open subset of an ALC need not be ALC

Example 2.13. Let X be the set of reals ,τ the topology on X as in example 2.13. The regular opens are those in τ_1 [10]. So X is ALC. Let $p \in X - Q$ and U be a regular open set containing p . Since every regular open set is open and $U \in \tau_1$ so there exists an open set V in X such that $p \in V \subseteq \overline{V} \subseteq U$, $\overline{V}$ compact. But $X - Q$,the open set of irrational, does not contain the closure of any of its open sets, since such closure must be identical to the usual Euclidean closure [10,Example 63]. Hence , $X - Q$ is not ALC .

Theorem 2.14. *Let X be ALC , $Y \subseteq X$. If Y is a regular open subset of X then Y is ALC.*

Proof. Let G be a regular open set in Y containing p , $p \in X$, i.e. $G = int(cl(G))$. Since G is regular open

$$G = int(cl_Y(G)) = int(cl_X(G)) \cap Y) = int(cl_X(G)) \cap intY$$

But $int(cl_X(G)) \cap Y = int(cl_X(G))$, since Y is regular open. Hence G is a regular open set in X. Since X is ALC then there exist an open set U and a nearly-compact set K such that

$$x \in U \subseteq K \subseteq G$$

Therefore, Y is ALC.

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