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SOME SUBORDINATION PROPERTIS FOR $p-$VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH LINEAR OPERATOR

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Abstract. In this paper, we obtain some subordination and superordination results of $p-$valent meromorphic functions associated with linear operator. Sandwich-type theorem for these multivalent function is also obtained .

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

For $f, F \in H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [12] and [13]).

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant

$\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [12] and [13]).
For functions $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ given by

$$g\left(z\right) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad \left(p \in \mathbb{N}\right), \tag{1.4}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$, is defined by

$$\left(f * g\right)\left(z\right) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k = \left(g * f\right)\left(z\right). \tag{1.5}$$

Aouf et al. [3] considered the following linear operator $D^n_{\lambda,p}\left(f * g\right)\left(z\right) : \sum_p \longrightarrow \sum_p$ as follows:

$$D^0_{\lambda,p}(f * g)(z) = (f * g)(z), \tag{1.6}$$

$$\begin{aligned} D^1_{\lambda,p}(f * g)(z) &= D_{\lambda,p}(f * g)(z) = (1-\lambda)(f * g)(z) + \frac{\lambda}{z^p}(z^{p+1}(f * g)(z))' \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]a_k b_k z^k \ (\lambda \geq 0;\ p \in \mathbb{N}), \end{aligned}$$

$$\begin{aligned} D^2_{\lambda,p}(f * g)(z) &= D_{\lambda,p}\left(D_{\lambda,p}(f * g)\right)(z) \\ &= (1-\lambda)D_{\lambda,p}(f * g)(z) + \frac{\lambda}{z^p}(z^{p+1}D_{\lambda,p}(f * g)(z))' \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^2 a_k b_k z^k \ (\lambda \geq 0;\ p \in \mathbb{N}), \end{aligned} \tag{1.7}$$

and (in general)
$D^n_{\lambda,p}(f*g)(z) = D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z))$

$$= \frac{1}{z^p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^n a_k b_k z^k \ (\lambda \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.8}$$

From (1.8) it is easy to verify that:

$$\lambda z(D^n_{\lambda,p}(f * g)(z))' = D^{n+1}_{\lambda,p}(f * g)(z) - (\lambda p + 1)D^n_{\lambda,p}(f * g)(z) \quad (\lambda > 0). \tag{1.9}$$

It should be remarked that the linear operator $D^m_{\lambda,p}(f * g)$ is a generalization of many other linear operators considered earlier. We have:
(1) If we take $g(z) = \frac{1}{z^p(1-z)}$ $(or\ b_k = 1)$, then we have the operator $D^n_{\lambda,p}(f)(z)$ which was introduced and studied by Aouf et al. [3];
(2) If we take $g(z) = \frac{1}{z^p(1-z)}$ $(or\ b_k = 1)$ and $\lambda = 1$, then we have the operator $M^n_p(f)(z)$ which was introduced and studied by Aouf and Hossen [2] and Srivastava and Patel [15];

(3) If we take $n=0$ and $g(z)=z^{-p}+\sum\limits_{k=0}^{\infty}\Psi_k(\alpha_1)z^k$ (*or* $b_k=\Psi_k(\alpha_1)$), where

$$\Psi_k(\alpha_1)=\frac{(\alpha_1)_{k+p}.......(\alpha_q)_{k+p}}{(\beta_1)_{k+p}...(\beta_s)_k\ (1)_{k+p}}\qquad (q\leq s+1;q,s\in\mathbb{N}_0)\,, \tag{1.10}$$

then the operator $D^0_{\lambda,p}\,(f*g)=(f*g)$ reduces to the operator $H_{p,q,s}\,(\alpha_1)$ which was introduced and studied by Liu and Srivastava [9]. The operator $H_{p,q,s}\,(\alpha_1)$ contains the operator $\ell_p(\alpha_1,\beta_1)$ [8] for $q=2,s=1$, and $\alpha_2=1$ and also contains the operator $D^{\nu+p-1}$ ([1], [4]) for $q=2,s=1$ and $\alpha_1=\nu+p$ $(\nu>-p;\ p\in\mathbb{N})$, $\alpha_2=1$ and $\beta_1=p$;

(4) If we take $n=0$ and $g(z)=z^{-p}+\sum\limits_{k=0}^{\infty}\left(\frac{l+\gamma(k+p)}{l}\right)^{\mu}z^k$

$(l>0,\ \gamma\geq 0,\ p\in\mathbb{N},\ \mu\in\mathbb{N}_0)$, then the operator $D^0_{\lambda,p}\,(f*g)=(f*g)$ reduces to the operator $J^{\mu}_p\,(\gamma,l)$ which was introduced and studied by El-Ashwah [5];

(5) If we take $n=0$ and $g(z)=z^{-p}+\sum\limits_{k=0}^{\infty}\left(\frac{l}{l+\gamma(k+p)}\right)^{\mu}z^k$

$(l>0,\ \gamma\geq 0,\ p\in\mathbb{N},\ \mu\in\mathbb{N}_0)$, then the operator $D^0_{\lambda,p}\,(f*g)=(f*g)$ reduces to the operator $\mathcal{L}^{\mu}_p\,(\gamma,l)$ which was introduced and studied by El-Ashwah [6].

(6) If we take $n=0$ and $g(z)=z^{-p}+\frac{\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\beta)}\sum\limits_{k=0}^{\infty}\left(\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k-\gamma+1)}\right)\frac{(\mu)_{k+p}}{(k+p)!}z^k$ $\mu>0,\ \beta>0,\ \alpha>\gamma-1,\ \gamma>0,p\in\mathbb{N}$, then the operator $D^0_{\lambda,p}\,(f*g)=(f*g)$ reduces to the operator $\mathcal{Q}^{p,\mu}_{\alpha,\beta,\gamma}$ which was introduced and studied by El-Ashwah et al. [7].

To prove our results, we need the following definitions and lemmas.

Definition 1.1 [12]. *Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U}\backslash E(q)$ where*

$$E(q)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\right\}$$

and are such that $q^{'}(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0)=a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0)\equiv\mathcal{F}_0$ and $\mathcal{F}(1)\equiv\mathcal{F}_1$.

Definition 1.2 [13]. *A function $L\,(z,t)$ $(z\in U,t\geq 0)$ is said to be a subordination chain if $L\,(0,t)$ is analytic and univalent in U for all $t\geq 0,L\,(z,0)$ is continuously differentiable on $[0;1)$ for all $z\in U$ and $L\,(z,t_1)\prec L\,(z,t_2)$ for all $0\leq t_1\leq t_2$.*

Lemma 1.1 [14]**.** *The function $L\,(z,t):U\times\ [0;1)\longrightarrow\mathbb{C}$ of the form*

$$L\,(z,t)=a_1\,(t)\,z+a_2\,(t)\,z^2+...\qquad (a_1\,(t)\neq 0;t\geq 0)$$

and $\lim\limits_{t\to\infty}|a_1\,(t)|=\infty$ is a subordination chain if and only if

$$\mathrm{Re}\left\{\frac{z\partial L\,(z,t)\,/\partial z}{\partial L\,(z,t)\,/\partial t}\right\}>0\qquad (z\in U,t\geq 0)\,.$$

Lemma 1.2 [10]. *Suppose that the function $\mathcal{H}:\mathbb{C}^2\to\mathbb{C}$ satisfies the condition*

$$\mathrm{Re}\left\{\mathcal{H}\,(is;t)\right\}\leq 0$$

for all real s *and for all* $t \leq -n\left(1+s^2\right)/2,\ n \in \mathbb{N}$. *If the function* $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$ *is analytic in* U *and*

$$\operatorname{Re}\left\{\mathcal{H}\left(p(z); zp^{'}(z)\right)\right\} > 0 \qquad (z \in U),$$

then $\operatorname{Re}\{p(z)\} > 0$ *for* $z \in U$.

Lemma 1.3 [11]. *Let* $\kappa, \gamma \in \mathbb{C}$ *with* $\kappa \neq 0$ *and let* $h \in \mathcal{H}(U)$ *with* $h(0) = c$. *If* $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0\,(z \in U)$, *then the solution of the following differential equation:*

$$q(z) + \frac{zq^{'}(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in U *and satisfies* $\operatorname{Re}\{\kappa q(z) + \gamma\} > 0$ *for* $z \in U$.

Lemma 1.4 [12]. *Let* $p \in \mathcal{F}(a)$ *and let* $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$*be analytic in* U *with* $q(z) \neq a$ *and* $n \geq 1$. *If* q *is not subordinate to* p, *then there exists two points* $z_0 = r_0 e^{i\theta} \in U$ *and* $\zeta_0 \in \partial U \backslash E(q)$ *such that*

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p^{'}(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Lemma 1.5 [13]. *Let* $q \in \mathcal{H}[a;1]$ *and* $\varphi : \mathbb{C}^2 \to \mathbb{C}$. *Also set* $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$. *If* $L(z,t) = \varphi\left(q(z), tzq^{'}(z)\right)$ *is a subordination chain and* $q \in H[a;1] \cap \mathcal{F}(a)$, *then*

$$h(z) \prec \varphi\left(q(z), zq^{'}(z)\right),$$

implies that $q(z) \prec p(z)$. *Furthermore, if* $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$ *has a univalent solution* $q \in \mathcal{F}(a)$, *then* q *is the best subordinant.*

In this paper, we investigate several properties of the linear operator $\mathcal{D}^n_{\lambda,p}(f * g)(z)$. $\blacksquare$

2. Main Results

Unless otherwise mentioned, we assume throughout this section that $\lambda, \alpha, \gamma > 0, p \in \mathbb{N}, n \in \mathbb{N}_0,\ z \in U$ and all powers are understood as principle values.

Theorem 2.1. *Let* $f, g, k, \psi \in \sum_p$ and let

$$\operatorname{Re}\left\{1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\} > -\delta$$

$$\left(\begin{array}{c} \phi(z) = (1-\gamma)\left(z^p \mathcal{D}^n_{\lambda,p}(k*\psi)(z)\right)^{\alpha} \\ +\gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(k*\psi)(z)}{\mathcal{D}^n_{\lambda,p}(k*\psi)(z)}\right)\left(z^p \mathcal{D}^n_{\lambda,p}(k*\psi)(z)\right)^{\alpha}; z \in U \end{array}\right), \tag{2.1}$$

where δ *is given by*

$$\delta = \frac{(\lambda\gamma)^2 + \alpha^2 - \left|(\lambda\gamma)^2 - \alpha^2\right|}{4\lambda\gamma\alpha}. \tag{2.2}$$

Then the subordination condition

$$(1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha+\gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)}{\mathcal{D}^n_{\lambda,p}(f*g)(z)}\right)\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha$$

$$\prec\quad(1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(k*\psi)(z)\right)^\alpha+\gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(k*\psi)(z)}{\mathcal{D}^n_{\lambda,p}(k*\psi)(z)}\right)\left(z^p\mathcal{D}^n_{\lambda,p}(k*\psi)(z)\right)^\alpha \tag{2.3}$$

implies that

$$\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha\prec\left(z^p\mathcal{D}^n_{\lambda,p}(k*\psi)(z)\right)^\alpha \tag{2.4}$$

and the function $\left(z^p\mathcal{D}^n_{\lambda,p}(k*\psi)\right)^\alpha$ *is the best dominant.*

Proof. Let us define the functions $F(z)$ and $G(z)$ in U by

$$F(z)=\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha\quad\text{and}\quad G(z)=\left(z^p\mathcal{D}^n_{\lambda,p}(k*\psi)(z)\right)^\alpha\quad(z\in U), \tag{2.5}$$

we assume here, without loss of generality, that $G(z)$ is analytic and univalent on $\bar{U}$ and

$$G^{'}(\zeta)\neq 0\qquad(|\zeta|=1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0<\rho<1$. These new functions have the desired properties on $\bar{U}$, so we can use them in the proof of our result, the results would follow by letting $\rho\to 1$.

We first show that, if

$$q(z)=1+\frac{zG^{''}(z)}{G^{'}(z)}\qquad(z\in U), \tag{2.6}$$

then

$$\operatorname{Re}\{q(z)\}>0\qquad(z\in U).$$

From (1.9) and the definition of the functions G,ϕ, we obtain that

$$\phi(z)=G(z)+\frac{\lambda\gamma}{\alpha}zG^{'}(z). \tag{2.7}$$

Differentiating both side of (2.7) with respect to z yields

$$\phi^{'}(z)=\left(1+\frac{\lambda\gamma}{\alpha}\right)G^{'}(z)+\frac{\lambda\gamma}{\alpha}zG^{''}(z). \tag{2.8}$$

Combining (2.6) and (2.8), we easily get

$$1+\frac{z\phi^{''}(z)}{\phi^{'}(z)}=q(z)+\frac{zq^{'}(z)}{q(z)+\frac{\alpha}{\lambda\gamma}}=h(z)\qquad(z\in U). \tag{2.9}$$

It follows from (2.1) and (2.9) that

$$\operatorname{Re}\left\{h(z)+\frac{\alpha}{\lambda\gamma}\right\}>0\quad(z\in U). \tag{2.10}$$

Moreover, by using Lemma 1.3, we conclude that the differential equation (2.9) has a solution $q(z)\in H(U)$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+\frac{\alpha}{\lambda\gamma}}+\delta,$$

where δ is given by (2.2). From (2.9) and (2.10), we obtain

$$\operatorname{Re}\left\{H\left(q(z); zq^{'}(z)\right)\right\} > 0 \qquad (z \in U).$$

To verify the condition that

$$\operatorname{Re}\left\{H\left(iu; v\right)\right\} \leq 0 \qquad \left(u \in \mathbb{R}; v \leq -\frac{1+u^2}{2}\right), \tag{2.11}$$

we proceed as follows:

$$\begin{aligned}\operatorname{Re}\left\{H\left(iu; v\right)\right\} &= \operatorname{Re}\left\{iu + \frac{v}{iu + \frac{\alpha}{\lambda\gamma}} + \delta\right\} = \frac{\frac{\alpha}{\lambda\gamma}v}{u^2 + \left(\frac{\alpha}{\lambda\gamma}\right)^2} + \delta \\ &\leq -\frac{\sigma\left(u, \lambda, \alpha, \delta\right)}{2\left[u^2 + \left(\frac{\alpha}{\lambda\gamma}\right)^2\right]},\end{aligned}$$

where

$$\sigma\left(u, \lambda, \alpha, \delta\right) = \left[\frac{\alpha}{\lambda\gamma} - 2\delta\right]s^2 - 2\delta\left(\frac{\alpha}{\lambda\gamma}\right)^2 + \frac{\alpha}{\lambda\gamma}. \tag{2.12}$$

For δ given by (2.2), we note that the expression $\sigma\left(u, \lambda, \alpha, \delta\right)$ in (2.12) is positive, which implies that (2.11) holds. Thus, by using Lemma 1.2, we conclude that

$$\operatorname{Re}\left\{q\left(z\right)\right\} > 0 \qquad (z \in U).$$

that is, that $G\left(z\right)$ defined by (2.5) is convex (univalent) in U. Next, we prove that the subordination condition (2.3) implies that

$$F\left(z\right) \prec G\left(z\right),$$

for the functions F and G defined by (2.5). Consider the function $L\left(z, t\right)$ given by

$$L\left(z, t\right) = G\left(z\right) + \frac{\lambda\gamma\left(1+t\right)}{\alpha}zG^{'}\left(z\right) \quad \left(0 \leq t < \infty; z \in U\right). \tag{2.13}$$

We note that

$$\left.\frac{\partial L\left(z, t\right)}{\partial z}\right|_{z=0} = G^{'}\left(0\right)\left(1 + \frac{\lambda\gamma\left(1+t\right)}{\alpha}\right) \neq 0 \qquad \left(0 \leq t < \infty; z \in U\right).$$

This show that the function

$$L\left(z, t\right) = a_1\left(t\right)z + \ldots,$$

satisfies the condition $a_1\left(t\right) \neq 0$ $\left(0 \leq t < \infty\right)$. Further, we have

$$\operatorname{Re}\left\{\frac{z\partial L\left(z, t\right)/\partial z}{\partial L\left(z, t\right)/\partial t}\right\} = \operatorname{Re}\left\{\frac{\alpha}{\lambda\gamma} + \left(1+t\right)q\left(z\right)\right\} > 0 \quad \left(0 \leq t < \infty; z \in U\right).$$

Therefore, by using Lemma 1.1, we deduce that $L\left(z, t\right)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi\left(z\right) = G\left(z\right) + \frac{\lambda\gamma}{\alpha}zG^{'}\left(z\right) = L\left(z, 0\right),$$

and

$$L\left(z, 0\right) \prec L\left(z, t\right) \quad \left(0 \leq t < \infty\right),$$

which implies that

$$L(\zeta,t)\notin L(U,0)=\phi(U)\quad(0\leq t<\infty;\zeta\in\partial U). \tag{2.14}$$

If F is not subordinate to G, by using Lemma 4, we know that there exist two points $z_0\in U$ and $\zeta_0\in\partial U$ such that

$$F(z_0)=G(\zeta_0)\quad\text{and}\quad z_0F^{'}(z_0)=(1+t)\zeta_0G^{'}(\zeta_0)\quad(0\leq t<\infty). \tag{2.15}$$

Hence, by virtue of (2.3), (2.5), (2.13) and (2.15), we have

$$\begin{aligned}L(\zeta_0,t) &= G(\zeta_0)+\frac{\lambda\gamma(1+t)zG^{'}(\zeta_0)}{\alpha}\\ &= F(z_0)+\frac{\lambda\gamma z_0F^{'}(z_0)}{\alpha}\\ &= (1-\gamma)\left(z_0^p\mathcal{D}_{\lambda,p}^n(f*g)(z_0)\right)^\alpha\end{aligned}$$

$$+\gamma\left(\frac{\mathcal{D}_{\lambda,p}^{n+1}(f*g)(z_0)}{\mathcal{D}_{\lambda,p}^n(f*g)(z_0)}\right)\left(z_0^p\mathcal{D}_{\lambda,p}^n(f*g)(z_0)\right)^\alpha\in\phi(U).$$

TThis contradicts (2.14). Thus, we deduce that $F\prec G$. Considering $F=G$, we see that the function G is the best dominant. This completes the proof of Theorem 2.1.

We now derive the following superordination result.

Theorem 2.2. *Let* $f,g,k,\psi\in\sum_p$ and let

$$\operatorname{Re}\left\{1+\frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\}>-\delta$$

$$\left(\begin{array}{c}\phi(z)=(1-\gamma)\left(z^p\mathcal{D}_{\lambda,p}^n(k*\psi)(z)\right)^\alpha\\ +\gamma\left(\frac{\mathcal{D}_{\lambda,p}^{n+1}(k*\psi)(z)}{\mathcal{D}_{\lambda,p}^n(k*\psi)(z)}\right)\left(z^p\mathcal{D}_{\lambda,p}^n(k*\psi)(z)\right)^\alpha\end{array}\right), \tag{2.16}$$

where δ *is given by* (2.2). *If the function*

$$(1-\gamma)\left(z^p\mathcal{D}_{\lambda,p}^n(f*g)(z)\right)^\alpha+\gamma\left(\frac{\mathcal{D}_{\lambda,p}^{n+1}(f*g)(z)}{\mathcal{D}_{\lambda,p}^n(f*g)(z)}\right)\left(z^p\mathcal{D}_{\lambda,p}^n(f*g)(z)\right)^\alpha$$

is univalent in U *and* $\left(z^p\mathcal{D}_{\lambda,p}^n(f*g)(z)\right)^\alpha\in\mathcal{F}$, *then the superordination condition*

$$\begin{aligned}&(1-\gamma)\left(z^p\mathcal{D}_{\lambda,p}^n(k*\psi)(z)\right)^\alpha+\gamma\left(\frac{\mathcal{D}_{\lambda,p}^{n+1}(k*\psi)(z)}{\mathcal{D}_{\lambda,p}^n(k*\psi)(z)}\right)\left(z^p\mathcal{D}_{\lambda,p}^n(k*\psi)(z)\right)^\alpha\\ \prec\ &(1-\gamma)\left(z^p\mathcal{D}_{\lambda,p}^n(f*g)(z)\right)^\alpha+\gamma\left(\frac{\mathcal{D}_{\lambda,p}^{n+1}(f*g)(z)}{\mathcal{D}_{\lambda,p}^n(f*g)(z)}\right)\left(z^p\mathcal{D}_{\lambda,p}^n(f*g)(z)\right)^\alpha\end{aligned}$$

implies that

$$\left(z^p\mathcal{D}_{\lambda,p}^n(k*\psi)(z)\right)^\alpha\prec\left(z^p\mathcal{D}_{\lambda,p}^n(f*g)(z)\right)^\alpha$$

and the function $\left(z^p\mathcal{D}_{\lambda,p}^n(k*\psi)(z)\right)^\alpha$ *is the best subordinant.*

Proof. Suppose that the functions F, G and q are defined by (2.5) and (2.6), respectively. By applying the similar method as in the proof of Theorem 2.1, we get

$$\operatorname{Re}\{q(z)\} > 0 \qquad (z \in U).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ be defined by (2.13). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{\lambda\gamma}{\alpha} zG'(z) = \varphi\left(G(z), zG'(z)\right)$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 2.2.

Combining the above-mentioned subordination and superordination results involving the operator $\mathcal{D}^n_{\lambda,p}(f*g)$, the following "sandwich-type result" is derived.

Theorem 2.3. *Let* $f, g, k_j, \psi_j \in \sum_p (j = 1, 2)$ and let

$$\operatorname{Re}\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta$$

$$\left(\begin{array}{c} \phi_j(z) = (1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(k_j*\psi_j)(z)\right)^\alpha \\ +\gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(k_j*\psi_j)(z)}{\mathcal{D}^n_{\lambda,p}(k_j*\psi_j)(z)}\right)\left(z^p\mathcal{D}^n_{\lambda,p}(k_j*\psi_j)(z)\right)^\alpha (j=1,2)\end{array}\right),$$

where δ *is given by* (2.2). *If the function*

$$(1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha + \gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)}{\mathcal{D}^n_{\lambda,p}(f*g)(z)}\right)\left(z^p\mathcal{D}^{n\theta}_{\lambda,p}(f*g)(z)\right)^\alpha$$

is univalent in U *and* $\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha \in \mathcal{F}$, *then the condition*

$$\begin{aligned}
&(1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(k_1*\psi_1)(z)\right)^\alpha + \gamma\frac{\mathcal{D}^{n+1}_{\lambda,p}(k_1*\psi_1)(z)}{\mathcal{D}^n_{\lambda,p}(k_1*\psi_1)(z)}\left(z^p\mathcal{D}^n_{\lambda,p}(k_1*\psi_1)(z)\right)^\alpha \\
\prec\ &(1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha + \gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)}{\mathcal{D}^n_{\lambda,p}(f*g)(z)}\right)\left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha \\
\prec\ &(1-\gamma)\left(z^p\mathcal{D}^n_{\lambda,p}(k_2*\psi_2)(z)\right)^\alpha + \gamma\left(\frac{\mathcal{D}^{n+1}_{\lambda,p}(k_2*\psi_2)(z)}{\mathcal{D}^n_{\lambda,p}(k_2*\psi_2)(z)}\right)\left(z^p\mathcal{D}^n_{\lambda,p}(k_2*\psi_2)(z)\right)^\alpha
\end{aligned}$$

implies that

$$\left(z^p\mathcal{D}^n_{\lambda,p}(k_1*\psi_1)(z)\right)^\alpha \prec \left(z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)^\alpha \prec \left(z^p\mathcal{D}^n_{\lambda,p}(k_2*\psi_2)(z)\right)^\alpha$$

and the functions $\left(z^p\mathcal{D}^\alpha_{\lambda,p}(k_1*\psi_1)(z)\right)^\mu$ *and* $\left(z^p\mathcal{D}^\alpha_{\lambda,p}(k_2*\psi_2)(z)\right)^\mu$ *are , respectively, the best subordinant and the best dominant.*

Remark 1. Specializing n, λ and $g(z)$ in the above results, we obtain the corresponding results for the corresponding operators (1-6) defined in the introduction.

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