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Generalisation of Z-transformation through complex variable Distribution theory

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Abstract

This paper provides a generalisation of the theory of Z-transformation Through Complex variable distribution theory. In this paper different properties of Testing function and Distribution are described. In this paper, we are able to formalise a generalised function and testing function of complex variable. Linear integral transformation of generalised function of complex variable is discussed. This paper provides the inversion formula and uniqueness theorem of generalised function of Z-transformation.

Keywords

Z-transformations, discrete time domain, testing function, generalised function etc.

1. Introduction

In Mathematical Sciences, Engineering, Physics and other fields of applied sciences, Various Forms of transforms such as Integral transforms, Laplace transforms, Fourier transforms etc are used depending on the problem whether discrete or continuous case. Discrete systems cannot be studied using Laplace transforms because they are continuous functions.

Z-transformation converts a sequence of real or complex numbers into a complex frequency representation .In this Paper, we construct Testing functions and generalised function of complex variable. We derive generalisation of Z-transformation through complex variable. We derive Inversion formula and uniqueness theorem of Z-transformation[1,2].

2. Testing Function[3]

A testing function $\psi(z)$ is a complex valued function of complex variable , is defined as

$$\psi(z,r)=\begin{cases} e^{-\frac{r^2}{r^2-z^2}}, |z|<r \\ 0, |z|\geq r \end{cases}$$

Where $\psi(z)$ can be differentiated an arbitrary number of times and which is identical to zero outside a finite interval.

2.1 Properties of Testing function

(i)If $f(z)$ can be differentiated arbitrary often $\phi(z)=f(z)\psi(z)$ =testing function.

(ii)If $f(z)$ is zero outside a finite interval $\phi(z)=\int_{\delta-j\infty}^{\delta+j\infty} f(\tau)\psi(z-\tau)d\tau, -\infty \prec z \prec \infty$ =testing function

Sequence of testing functions $\{\psi_n\}, 1<n<\infty$ converges to zero if all ψ_n are identically zero outside some interval independent of n and each ψ_n as well as all of its derivatives, tends uniformly to zero.

3 Distributions[3]

Distribution (or Generalised function) $g(z)$ is a process of assigning an arbitrary function $\psi(z)$ to a number $N_g[\psi(z)]$.Distribution is a functional.

An ordinary function $g(z)$ is a distribution if $\int_{\delta-j\infty}^{\delta+j\infty} g(z)\psi(z)dz = N_g[\psi(z)]$ exists for every test function $\psi(z)$ in the set. Also, If $g(z)=f(z)$ then $\int_{\delta-j\infty}^{\delta+j\infty} f(z)\psi(z)dz=\int_0^{\infty}\psi(z)dz$ The function $f(z)$ is a distribution that assigns to $\psi(z)$ a number equal to its area from zero to infinity.

3.1Properties of Distributions

(i) Linearity-homogeneity

$$\int_{\delta-j\infty}^{\delta+j\infty} g(z)[a_1\psi_1(z)+a_2\psi_2(z)]dz = a_1\int_{\delta-j\infty}^{\delta+j\infty} g(z)\psi_1(z)dz + a_2\int_{\delta-j\infty}^{\delta+j\infty} g(z)\psi_2(z)dz$$

(ii)Summation

$$\int_{\delta-j\infty}^{\delta+j\infty} [g_1(z)+g_2(z)]\psi(z)dz = \int_{\delta-j\infty}^{\delta+j\infty} g_1(z)\psi(z)dz + \int_{\delta-j\infty}^{\delta+j\infty} g_2(z)\psi(z)dz$$

(iii)Shifting

$$\int_{\delta-j\infty}^{\delta+j\infty} g(z-z_0)\psi(z)dz = \int_{-\infty}^{\infty} g(z)\psi(z+z_0)dz$$

(iv) Scaling

$$\int_{\delta-j\infty}^{\delta+j\infty} g(az)\psi(z)dz = \frac{1}{|a|}\int_{-\infty}^{\infty} g(z)\psi(\frac{z}{a})dz$$

(v)Even distribution $\int_{\delta-j\infty}^{\delta+j\infty} g(z)\psi(z)dz = 0, \psi(z)=odd$

(vi)Odd distribution

$$\int_{\delta-j\infty}^{\delta+j\infty} g(z)\psi(z)dz = 0, \psi(z) = even$$

(vii)Derivative

$$\int_{\delta-j\infty}^{\delta+j\infty} \frac{dg(z)}{dz}\psi(z)dz = g(z)\psi(z)\Big|_{\delta-j\infty}^{\delta+j\infty} - \int_{\delta-j\infty}^{\delta+j\infty} g(z)\frac{d\psi(z)}{dt}dt = -\int_{\delta-j\infty}^{\delta+j\infty} g(z)\frac{d\psi(z)}{dz}dz$$

Where the integrand term is equal to zero in view of the properties of testing function.

(viii)The nth derivative

$$\int_{\delta-j\infty}^{\delta+j\infty} \frac{d^n g(z)}{dz^n}\psi(z)dz = (-1)^n \int_{\delta-j\infty}^{\delta+j\infty} g(z)\frac{d^n\psi(z)}{dz^n}dz$$

(ix)Product with ordinary function

$$\int_{\delta-j\infty}^{\delta+j\infty} [g(z)f(z)]\psi(z)dz = \int_{\delta-j\infty}^{\delta+j\infty} g(z)[f(z)\psi(z)]dz$$

Provided $f(z)\psi(z)$ belongs to the set of test functions.

(x)Convolution

$$\int_{\Sigma-j\infty}^{\Sigma+j\infty} [\int_{\Sigma-j\infty}^{\Sigma+j\infty} g_1(w)g_2(z-w)dw]\psi(z)dz = \int_{\Sigma-j\infty}^{\Sigma+j\infty} g_1(w)[\int_{\Sigma-j\infty}^{\Sigma+j\infty} g_2(z-w)\psi(z)dz]dw$$

By formal change of the order of Integration.

4 function transformation of Complex variable[4,5,6]

Consider A be a set of functions f defined on a subset Ω of C^n and B be a non empty set. A function transformation T is defined as a mapping from A to B. The image of f under T, which will be denoted by $\mathrm{T}[f]$ or $(\mathrm{T}f)$ is called the T -transform of f .The set A is called the domain of T and the set $\{T(f): f \in A\}$ is called the domain of T and the set $\{T(f): f \in A\}$ is called the range. The set B is called the co-domain or the target space of T. If A is a set of sequence of complex numbers, the T is said to be a sequence transformation .The image of sequence $\{\mathrm{a}_n\}$ under T is denoted by $T[\{a_n\}]$.

A function transformation of complex variable is called an integral transformation of complex variable if there exists a pair of functions G and K such that $T|f|(w) = \int_{\Omega_1} G(z, w, f(z), K(z,w))dz$ where Ω_1 is a measurable subset of Ω, $z \in c^n, w \in \Omega_2 \subseteq c^m$ and the integral is assumed to exist for every $f \in A$.

5 Linear Integral Transformation of Generalized function[4,5,6]

We have

$$T[f](w) = \int_I f(z)K(z,w)dz.$$

If the Kernal of the transformation is an infinitely differentiable function in z a testing –function space V(Ω) is connected so that for each fixed w in same region Ω^* in c^m.

The transform of any $f \in V^*(\Omega)$ is defined by $T[f](w) =< f(z), K(z,w) >$,where $< f, K >$ denotes the number that the linear functional f assigns to K as a function of z .The definition of T ,of course, depends on how the space V is constructed.

6 Distribution or Generalised function in discrete time domain[4,5,6]

A distribution $G(z)$ may be defined as the value of the integral $J_G[\psi(z)]$ of its product with a test function $\psi(z)$,

Symbolically, we write $J_G[\psi(z)] = < \mathrm{G(z)}\psi(z) >_{|z|=r} = \oint_{|z|=r} G(z)\psi(z)dz$

Where the contour of Integration is a circle of radius $r = |z|$ centered at the origin in the z-plane.

6.1 Discrete transform of Generalised function $J_G[\psi(z)]$

Consider J be an interval and R be the Linear differential and R be the linear differential operator. Assume that there exists a sequence $\{\lambda_n\}^{\infty}{}_{n=0}$ of real eigen values of R and a sequence $\{J_{G_n}[\psi(z)]\}^{\infty}{}_{n=0}$ of infinitely differentiable eigen functions on J such that $RJ_{G_n}[\psi(z)] = \lambda_n J_{G_n}[\psi(z)]$

With $\{J_{G_n}[\psi(z)]\}^{\infty}{}_{n=0}$ being a complete orthonormal family on $T^2(J)$.Consider A as the set of all complex -valued c^{∞} -function $J_G[\psi(z)]$ on J that satisfy the following two conditions

(i) $\gamma_K(J_G[\psi(z)]) = [\int_J |(R^K J_G[\psi(z)|^2 dz]^{\frac{1}{2}} < \infty$ for $k = 0,1,2,\ldots\ldots\ldots\ldots$

(ii) $< R^K, J_G[\psi(z)], J_{G_n}[\psi(z)] >=< J_G[\psi(z)], \mathrm{R}^K, J_{G_n}[\psi(z)]$ for each n and K.

It is not difficult to show that A is a Linear subspaces of $T^2(J)$ and that $\{\gamma_K\}_{K=0}^{\infty}$ is a separating family of semi norms. Hence A is count ably normed space .In fact, it is a Frechet space. A is a subspace of $\xi(J)$ and that the convergence in A implies uniform convergence on compact subset of J.Therefore A is a testing –function space. The eigen functions $\{J_{G_m}[\psi(z)]\}_{m=0}^{\infty}$ belongs to A,they are in $c^{\infty}(J)$ and in addition they satisfy conditions(i) and (ii) mentioned above.

7 Inversion formula of generalized function $J_{G_n}[\psi(z)$ in A^*

$$\gamma_K(J_{G_m}[\psi(z)]) = |\lambda_m|^k [\int_J |J_{G_m}[\psi(z)|^2 dz]^{\frac{1}{2}} = |\lambda_m|^K < \infty, \text{For } k = 0,1,2,\ldots\ldots\ldots\ldots$$

Furthermore,

we have $\quad < R^K, J_{G_m}[\psi(z)], J_{G_n}[\psi(z)] >= \lambda_m^K < J_{G_m}[\psi(z)], J_{G_n}[\psi(z)] >= \lambda_m^K \delta_{m,n},$

and because the eigen values are real, we have

$< J_{G_m}[\psi(z)], \mathrm{R}^K J_{G_n}[\psi(z)] >= \lambda_n^K < J_{G_m}[\psi(z)], J_{G_n}[\psi(z)] >= \lambda_n^K \delta_{m,n},$

Therefore $< R^K, J_{G_m}[\psi(z)], J_{G_n}[\psi(z)] >=< J_{G_m}[\psi(z)], \mathrm{R}^K J_{G_n}[\psi(z)] >$ for all m and n.

Since $\{J_{G_n}[\psi(z)]\}_{n=0}^{\infty} \subseteq A$,the numbers $< f, J_{G_n}[\psi(z)] >$ are well defined and hence we can define the transformation T on A^* by $T|f| = \{< f, J_{G_n}[\psi(z)] >\}_{n=0}^{\infty}$ If $J_G[\psi(z)] \in \mathrm{A}$

then $\phi = \sum_{n=0}^{\infty} <J_G[\psi(z)], J_{G_n}[\psi(z)]> J_{G_n}[\psi(z)]$, where the series converges in the sense of A .Moreover, If $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers, Then the series $\sum_{n=0}^{\infty} a_n J_{G_n}[\psi(z)]$ converges in A if and only if $\sum_{n=0}^{\infty} |\lambda_n|^{2k} |a_n|^2 < \infty$ for every non-negative integer k.

7.1 Theorem[7]

If $f \in A^*$, then $f = \sum_{n=0}^{\infty} < f, J_{G_n}[\psi(z)] > J_{G_n}[\psi(z)]$

Where the series converges in A^*.

Proof. Consider if $\{b_n\}_{n=0}^{\infty}$ is a sequence of complex numbers, then the series $\sum_{n=0}^{\infty} b_n J_{G_n}[\psi(z)]$ converges in A^* if and only if there exists a non-negative integer q such that $\sum_{\lambda_n \neq 0} |\lambda_n|^{-2q} |b_n|^2 < \infty$

Furthermore, if we denote the sum of the series $\sum_{n=0}^{\infty} b_n J_{G_n}[\psi(z)]$ by f then $b_n =< f, J_{G_n}[\psi(z)] >$

Finally the transformation T is the one to one and continuous that is,if $T[f] = T[g]$ then $f = g$

In A^* and if $\lim_{v \to \infty} f_v = f$ in A^* then

$$\lim_{v \to \infty} T[f_v] = \lim_{v \to \infty} \{< f_v, J_{G_n}[\psi(z)] >\}_{n=0}^{\infty} = \{< f, J_{G_n}[\psi(z)] >\}_{n=0}^{\infty} = T[f]$$

Hence the series converges in A^* for every $J_G[\psi(z)] \in A$.

7.2 Inversion formula[7]

The inversion formula for a certain generalized integral transformation T ,defined by $Tf \square F(n) \square < f, J_{G_n}[\psi(z)] >$ $\quad f \in A^*$; Thus T is a mapping of A^* into the space of complex-valued functions defined on n .The inverse mapping T^{-1} is given as $T^{-1}F(n) = \sum_{n=0}^{\infty} F(n) J_{G_n}[\psi(z)] = f$

7.3 The Uniqueness Theorem[7]

If $f, g \in A^*$ and if their transforms satisfy $F(n) = G(n)$ for every n, then $f = g$ in the sense of equality in A^*.

Proof- $f - g = \sum < f - g, J_{G_n}[\psi(z)] > J_{G_n}[\psi(z)]$

$= \sum [< f, J_{G_n}[\psi(z)] > - < g, J_{G_n}[\psi(z)] >] J_{G_n}[\psi(z)] = 0 \Rightarrow f = g.$

7.4 Conclusions

Generalization of z-transformation through complex variable distribution theory is achieved.

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