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NEW MATHEMATICAL MODELS OF DIVERGENCE MEASURES FOR FUZZY DISTRIBUTIONS

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ABSTRACT

Knowingly that distance measures are very important and play a significant role in application towards a variety of disciplines Engineering Sciences, we try to extend the same concept for the fuzzy distributions. The present communication is a step in that direction deriving new measures of fuzzy directed divergence and the study of their detailed properties.

Keywords: Directed divergence, Fuzzy distribution, Fuzzy divergence, Fuzzy set, Convex function.

INTRODUCTION

It is well known that in the literature of distance measures in the probability spaces, one such a measure which is most important and useful measure of directed divergence is due to Kullback and Leibler [9] and is given by

$$D(P:Q)=\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \tag{1.1}$$

In different disciplines, the concept of distance has been proved to be very important because of its applications areas but it seems to be natural that one should extend the concept of distance for applications to problems in other disciplines of social sciences, economics, biological sciences and other fields of environmental sciences.

In case of fuzzy distributions, we extend the concept of distance measure in such a manner that it describes the difference between fuzzy sets and can be considered as a dual concept of similarity measure. Many researchers, such as Yager [17] and Kaufmann [8] had used distance measure to define fuzzy entropy. Rosenfeld [14] defined the shortest distance between two fuzzy sets as a density function on the non - negative reals, which generalizes the definition of shortest distance for crisp sets in a natural way. Thus, corresponding to the probabilistic measure of divergence due to Kullback and Leibler [9], Bhandari and Pal [2] introduced the following measure of fuzzy directed divergence:

$$I(A:B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right] \tag{1.2}$$

Corresponding to Renyi's [13] and Havrada and Charvat's [4] probabilistic divergence measures, Kapur [7] took the following expressions of fuzzy divergence measures:

$$D_\alpha(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n} \log\left[\ \mu_A^{\alpha}(x_i)\ \mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\ \right] \qquad (1.3)$$

and

$$D^\alpha(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\ \mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\ (1-\mu_B(x_i))^{1-\alpha} - 1\right] \qquad (1.4)$$

Parkash [11] introduced a new generalized measure of fuzzy directed divergence involving two real parameters, given by

$${}_1D_\alpha^\beta(A,B) = [(\alpha-1)\ \beta]^{-1}\sum_{i=1}^{n}\left[\left\{\ \mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\right\}^{\beta} - 1\right];$$

$$\alpha \neq 1, \alpha > 0, \beta \neq 0 \qquad (1.5)$$

In fact, Kapur [7] has developed many expressions of generalized measures of fuzzy directed divergence corresponding to probabilistic measures of divergence due to Harvada and Charvat [4], Renyi [13], Kapur [6], Sharma and Taneja [16] etc. Since, we are presently dealing with fuzzy distributions; we take into consideration the notion of fuzzy sets introduced by Zadeh [18]. Some other contributors towards the development of fuzzy divergence measures are Bhandari, Pal and Majumde [3], Lowen [10], Fan, Ma and Xie [5] etc. Taking into consideration the idea of fuzzy sets, we have developed a new non-parametric measure of fuzzy directed divergence.

2. NEW GENERALIZED MEASURES OF FUZZY DIRECTED DIVERGENCE FOR DISCRETE FUZZY DISTRIBUTIONS

In this section, we have proposed the following new measures of fuzzy directed divergence for the discrete probability distributions:

(I) Generalized Fuzzy Divergence Corresponding to Burg's [1] Divergence Measure:

We propose the following generalized expression for fuzzy directed divergence measure corresponding is Burg's [1] probabilistic measure of divergence:

$$D_1(A:B) = \sum_{i=1}^{n}\left[\frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} - \log\frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} + \frac{1+a\left(1-\mu_A(x_i)\right)}{1+a\left(1-\mu_B(x_i)\right)} - \log\frac{1+a\left(1-\mu_A(x_i)\right)}{1+a\left(1-\mu_B(x_i)\right)} - 2\right]; a > 0 \qquad (2.1)$$

The measure (2.1) is a more generalized form of the measure of fuzzy divergence developed by Parkash and Sharma [12].

To check the validity of the proposed measure, we study its following properties:

I. Non-negativity:

Equation (2.1) can be rewritten as

$$D_1(A:B) = Y + Z \tag{2.2}$$

where

$$Y = \sum_{i=1}^{n}\left[\frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} - \log\frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} - 1\right] \tag{2.3}$$

and

$$Z = \sum_{i=1}^{n}\left[\frac{1+a\left(1-\mu_A(x_i)\right)}{1+a\left(1-\mu_B(x_i)\right)} - \log\frac{1+a\left(1-\mu_A(x_i)\right)}{1+a\left(1-\mu_B(x_i)\right)} - 1\right] \tag{2.4}$$

Next, we take

$$Y = x - \log x - 1 \tag{2.5}$$

where

$$x = \frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)} \tag{2.6}$$

Since $x - \log x - 1 \begin{cases} > 0 & for\ x > 0 \\ = 0 & for\ x = 1 \end{cases}$

Thus, we see that $Y \geq 0$ and vanishes iff $\mu_A(x_i) = \mu_B(x_i)\ \ \forall i$

Similarly, by taking

$$Z = x' - \log x' - 1 \tag{2.7}$$

where

$$x' = \frac{1+a(1-\mu_A(x_i))}{1+a(1-\mu_B(x_i))}, \tag{2.8}$$

We can easily show that $Z \geq 0$ and vanishes iff $\mu_A(x_i) = \mu_B(x_i)\ \forall i$

Hence, equation (2.2) gives that

$$D_1(A:B) = Y + Z \geq 0$$

Thus $D_1(A:B)$ ≥ 0 and the equality holds only when *A*=*B*.

II. Convexity: We have

$$\frac{dY}{dx} = 1 - \frac{1}{x} \quad \text{and}$$

$$\frac{d^2Y}{dx^2} = \frac{1}{x^2} > 0$$

which shows that Y is a convex function of $\mu_A(x_i)$ *and* $\mu_B(x_i)$ $\forall i$. Similarly, it can be shown that Z is a convex function of $\mu_A(x_i)$ *and* $\mu_B(x_i)$. Thus, we have:

1. $D_1(A:B) \geq 0$
2. $D_1(A:B)$ is a convex function of both $\mu_A(x_i)$ and $\mu_B(x_i)$ $\forall$i.
3. $D_1(A:B)$ does not change when $\mu_A(x_i)$ is replaced by 1- $\mu_A(x_i)$ and $\mu_B(x_i)$ is replaced by 1- $\mu_B(x_i)$.
4. $D_1(A:B) = 0$ when $A = B$.

Under the above four conditions, the proposed measure $D_1(A:B)$ is a valid measure of weighted fuzzy divergence.

Note: We have $D^{'}(A,B) =$

$$\lim_{a\to\infty} D_1(A:B) = \sum_{i=1}^{n}\left[\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)} - \log\frac{\mu_A(x_i)}{\mu_B(x_i)} - 1\right\} + \left\{\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)} - \log\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)} - 1\right\}\right] \quad (2.9)$$

which is another measure of divergence introduced by Parkash and Sharma [12].

(II) Two Generalized Fuzzy Divergence Measures Corresponding to Parkash and Sharma's [12] Divergence Measure:

(a)Consider ${}_1D_\lambda(A:B) =$

$$\sum_{i=1}^{n}\left[\begin{array}{l}\dfrac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)} - \log\dfrac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)} \\ + \dfrac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))} - \log\dfrac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_A(x_i))} - 2\end{array}\right] \quad (2.10)$$

If we put λ = 0 in (2.10), we get ${}_1D_0(A:B) = D^{'}(A:B)$.

(b) Consider ${}_2D_\lambda(A:B) =$

$$\sum_{i=1}^{n}\left[\frac{1+a\mu_A(x_i)}{\lambda(1+a\mu_A(x_i))+(1-\lambda)(1+a\mu_B(x_i))}-\log\frac{(1+a\mu_A(x_i))}{\lambda(1+a\mu_A(x_i))+(1-\lambda)(1+a\mu_B(x_i))}\right]$$

$$+\sum_{i=1}^{n}\left[\frac{1+a(1-\mu_A(x_i))}{\lambda(1+a(1-\mu_A(x_i)))+(1-\lambda)(1+a(1-\mu_B(x_i)))}\right] \quad (2.11)$$

$$-\sum_{i=1}^{n}\left[\log\frac{1+a(1-\mu_A(x_i))}{\lambda(1+a(1-\mu_A(x_i)))+(1-\lambda)(1+a(1-\mu_B(x_i)))}-2\right]$$

Again, if we put λ = 0 in (2.11), we get the following result:

${}_2D_0(A:B)=D^{'}(A:B)$.

It can easily be verified that the generalized measures introduced in (2.10) and (2.11) are valid measures of fuzzy directed divergence.

Note: The monotonic character of the above parametric divergence measures can be discussed as below:

(I) Differentiating (2.10) with respect to λ, we get

$$\frac{d\,{}_1D_\lambda(A:B)}{d\lambda}=\sum_{i=1}^{n}\frac{(\lambda-1)\{\mu_A(x_i)-\mu_B(x_i)\}^2}{\{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)\}\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}}$$

$$< 0$$

Thus, we see that the weighted fuzzy measure ${}_1D_\lambda(A,B)$ is monotonically decreasing function of λ, 0< λ<1.

To check the decreasing behavior of the divergence measure ${}_1D_\lambda(A,B)$, we have computed different values of ${}_1D_\lambda(A,B)$ with the help of different parameters and different fuzzy values and thus obtained Fig-2.1.

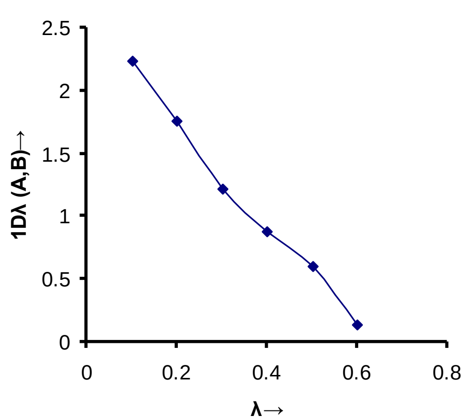

Fig-2.1

(II) Differentiating (2.11) with respect to λ, we again see that $\frac{d\,{}_2D_\lambda(A:B)}{d\lambda} < 0$

Thus, we see that ${}_2D_\lambda(A,B)$ is monotonically decreasing function of λ. Similarly, we have checked the decreasing behavior of ${}_2D_\lambda(A,B)$ as presented in Fig.-2.2.

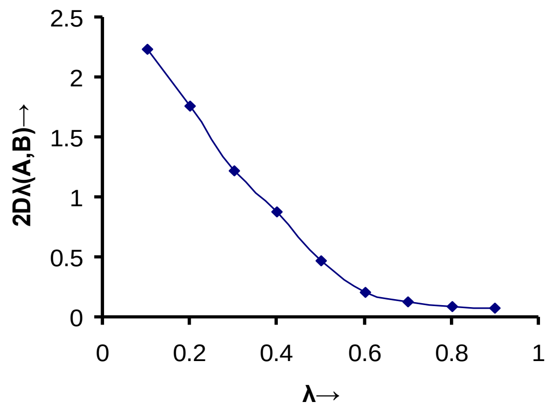

Fig-2.2

(III) Measure of Fuzzy Divergence Corresponding to Jensen-Shannon Measure and its Generalization:

We take the following expression for this measure:

$${}^{\lambda}D_1(A:B) =$$

$$\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\right] \quad (2.12)$$

We, prove that the weighted fuzzy measure (2.12) is a convex function of $\mu_A(x_1),\mu_A(x_2),...,\mu_A(x_n)$ and $\mu_B(x_1),\mu_B(x_2),...,\mu_B(x_n)$. The other properties for being a divergence measure are obvious. Next, rewriting equation (2.12), we have the following expression for the divergence measure:

$${}^{\lambda}D_1(A:B) =$$

$$-\sum_{i=1}^{n}\left[\mu_A(x_i)\log\{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)\}+(1-\mu_A(x_i))\log\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}\right]$$

$$+\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i)+(1-\mu_A(x_i))\log(1-\mu_A(x_i))\right] \quad (2.13)$$

Differentiating (2.13) w.r.t. $\mu_B(x_i)$, we get

$$\frac{d}{d\mu_B(x_i)}\,{}^{\lambda}D_1(A:B) = -\frac{(1-\lambda)\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)} + \frac{(1-\lambda)(1-\mu_A(x_i))}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}$$

Also, we have

$$\frac{\partial^2\,{}^{\lambda}D_1(A:B)}{\partial\mu_B^2(x_i)} = \frac{(1-\lambda)^2\mu_A(x_i)}{[\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)]^2} + \frac{(1-\lambda)^2(1-\mu_A(x_i))}{[\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))]^2} > 0$$

Thus ${}^{\lambda}D_1(A:B)$ is a convex function of $\mu_B(x_i)$. Similarly, we can prove that it is convex function of $\mu_A(x_i)$.

The above directed divergence can be generalized as follows:

$D_\lambda(A:B) =$

$$\lambda\sum_{i-1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\right]$$

$$+(1-\lambda)\sum_{i=1}^{n}\begin{bmatrix}\mu_B(x_i)\log\dfrac{\mu_B(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\\ \\ +(1-\mu_B(x_i))\log\dfrac{1-\mu_B(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\end{bmatrix} \tag{2.14}$$

$$= \lambda\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i)+(1-\mu_A(x_i))\log(1-\mu_A(x_i))\right] +$$

$$(1-\lambda)\sum_{i=1}^{n}\left[\mu_B(x_i)\log\mu_B(x_i)+(1-\mu_B(x_i))\log(1-\mu_B(x_i))\right]$$

$$-\sum_{i=1}^{n}\left[(\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i))\log\{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i))\}\right]$$

$$-\sum_{i=1}^{n}\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}\log\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}$$

$$= H(\lambda A+(1-\lambda)B)-\lambda H(A)-(1-\lambda)H(B)$$

where $H(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i)+(1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$

and it represents measure of fuzzy entropy corresponding to Shannon's [15] entropy. The measure (2.14) is a generalization of Parkash and Sharma's [12] measure of fuzzy divergence which arises when $\lambda = \frac{1}{2}$.

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