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ON FACTORIZATION OF ENTIRE ALGEBROIDAL FUNCTIONS

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ABSTRACT

In this paper we introduce the concept of primeness and pseudo-primeness in divisor sense for entire algebroidal functions. We also show that prime entire algebroidal functions may not be prime in divisor sense and vice versa.

Key words: algebroidal functions, entire functions, primeness in divisor sense, factorization.

AMS Subject Classification: 30D05.

1. INTRODUCTION AND PRELIMINARIES

Let W(z) be a k-valued function defined by the following irreducible equation:

$$A_k(z)\, W^k(z) + A_{k-1}(z)\, W^{k-1}(z) + \ldots + A_0(z) = 0 \ \ldots \qquad (1.1)$$

where $A_k(z) \not\equiv 0$, all $A_i(z)$, (i = 0, 1, 2, … , k) are entire functions and have no common zeros.

If at least one of $A_i(z)$ (i = 0, 1, … , k) is transcendental then W(z) is called k-valued strictly algebroidal function. Further, if $A_k(z) \equiv 1$, then W(z) is called k-valued entire strictly algebrodial function. If all the $A_i(z)$ (i = 0, 1,…, k) are polynomials, at least one of which is non-constant, then W(z) is called a k-valued algebraic function. If all the $A_i(z)$ (i = 0, 1, …, k) are at most linear, at least one of which is non-constant, then W(z) is called a k-valued linear algebraic function.

Throughout this paper, we call W(z) an algebroidal function if it is either strictly algebroidal or algebraic or linear algebraic unless otherwise stated.

We next give definition of zeros of some order of algebroidal functions.

Definition 1.1: Let W(z) be a k-valued algebroidal function. Then z_0 is called a zero of W(z) of order n if there exists a positive integer m such that

$$\left[\left|\frac{m}{k}\right|\right] = n \quad \left(\left[\left|\frac{m}{k}\right|\right] \text{ denotes the smallest integer} \geq x\right)$$

and $W(z) = (z - z_0)^{\frac{m}{k}}\, W_0(z)$, where $W_0(z)$ is at most k-valued algebroidal function with $W_0(z_0) \neq 0$.

We now state some fundamental concepts of factorization of algebroidal functions as extensions of meromorphic functions case (Sec [1], [2]).

Definition 1.2: Let W(z) be a k-valued algebroidal function, which can be expressed as

$$W(z) = W_0\,(g(z)) \qquad (1.2)$$

where $W_0(z)$ is a k_0-valued algebroidal function, $k_0 \leq k$, g(z) is an entire function (if $W_0(z)$ is a k_0-valued algebraic function, g(z) can be a meromorphic function). Then (1.2) is called a composite factorization of W(z). $W_0(z)$ and g(z) are called left and right factors of W(z), respectively.

Definition 1.3 Let W(z) be a k-valued algebroidal function. If every factorization of W(z) of the form (1.2) must imply that either g(z) is linear (respectively g(z) is a polynomial) or $W_0(z)$ is a k_0-valued linear algebraic function (respectively $W_0(z)$ is a k_0-valued algebraic function), then W(z) is called prime (respectively pseudo-prime). If every factorization of W(z) of the form (1.2) either implies that g(z) is a polynomial or $W_0(z)$ is a k_0-valued linear algebraic function, then W(z) is called left-prime.

Definition 1.4: Let W(z) be a k-valued algebroidal function defined by (1.1).

(i) If $\frac{A_i(z)}{A_k(z)}$ (i = 0, 1, …, k – 1) have no non-linear (respectively transcendental) entire function as common right factor or the only possible left factors of $\frac{A_i(z)}{A_k(z)}$ (i = 0, 1, …, k – 1) are bilinear, then W(z) is called prime (respectively left-prime).

(ii) If $\frac{A_i(z)}{A_k(z)}$ (i = 0, 1, …, k – 1) have no transcendental meromorphic function as common right factor or the only possible left factors of $\frac{A_i(z)}{A_k(z)}$ (i = 0, 1, …, k – 1) are rational functions, then W(z) is called pseudo-prime.

We now extend the definition of primeness and pseudo-primeness in divisor sense of entire functions (see [3], [4], [5], [6]) to entire algebroidal functions.

Definition 1.5: Let W(z) be a k-valued entire algebroidal function with zeros. If W(z) can be expressed as

$$W(z) = W_0(g(z))\, e^{\square(z)}, \qquad (1.3)$$

where $W_0(z)$ is k_0-valued entire algebroidal function, $k_0 \leq k$, g(z) ($\not\equiv$ constant) and $\square(z)$ are entire functions, then the expression (1.3) is called a composite factorization of W(z) in divisor sense. $W_0(z)$ and g(z) are called left and right factors, respectively, of W(z) in divisor sense.

Definition 1.6: Let W(z) be a k-valued entire algebroidal function with zeros. If every factorization of W(z) of the form (1.3) implies that

(i) $W_0(z)$ has just one simple zero or g(z) is a linear polynomial, then W(z) is called prime in divisor sense.
(ii) $W_0(z)$ has only a finite number of zeros or g(z) is a polynomial, then W(z) is called pseudo-prime in divisor sense.
(iii) g(z) is a linear polynomial whenever $W_0(z)$ has infinite number of zeros, then W(z) is called right prime in divisor sense.
(iv) $W_0(z)$ has just one simple zero whenever g(z) is transcendental, then W(z) is called left-prime in divisor sense.

MAIN RESULTS

In this section we prove the main results of the paper.

Theorem 2.1: Let $W(z)$ be a k-valued algebroidal function. Then z_0 is a zero of $W(z)$ of order n if and only if $W(z_0) = W'(z_0) = \cdots = W^{(n-1)}(z_0) = 0$ and $W^{(n)}(z_0) \neq 0$.

Proof: First suppose z_0 is a zero of W of order n. Then there exists a positive integer m such that $\left[\left|\frac{m}{k}\right|\right] = n$ and $W(z) = (z - z_0)^{\frac{m}{k}} W_0(z)$, where $W_0(z_0) \neq 0$.

Case (i): Let $q = \frac{m}{k}$ be a non-integer. Then

$$W'(z) = q(z - z_0)^{q-1} W_0(z) + (z - z_0)^q W_0'(z)$$
$$= (z - z_0)^{q-1}[q W_0(z) + (z - z_0) W_0'(z)]$$
$$= (z - z_0)^{q-1} \phi_1(z),$$

where $\phi_1(z_0) \neq 0$. Thus, $W'(z_0) = 0$

Continuing, we get $W^{([q])}(z) = (z - z_0)^{q-[q]} \phi_{[q]}(z)$ ($[x]$ denotes greatest integer $\leq x$),

where $\phi_{[q]}(z_0) \neq 0$ and hence $W^{([q])}(z_0) = 0$, whereas $W^{([q]+1)}(z) = \frac{1}{(z - z_0)^{[q]+1-q}} \phi_{[q]+1}(z)$

and so $W^{([q]+1)}(z_0) \neq 0$.

Thus, $W(z_0) = W'(z_0) = \cdots = W^{([q])}(z_0) = 0$ and $W^{([q]+1)}(z_0) \neq 0$.

So, $W(z_0) = W'(z_0) = W^{([|q|]-1)}(z_0) = 0$ and $W^{[|q|]}(z_0) \neq 0$.

Hence, $W(z_0) = W'(z_0) = \cdots = W^{(n-1)}(z_0) = 0$ and $W^{(n)}(z_0) \neq 0$.

Case (ii): Let $q = \frac{m}{k}$ be an integer. Then, since $q = [q] = [|q|]$, we easily conclude that

$W(z_0) = W'(z_0) = \cdots = W^{(q-1)}(z_0) = 0$ and $W^{(q)} \neq 0$

and so $W(z_0) = W'(z_0) = \cdots = W^{(n-1)}(z_0) = 0$ and $W^{(n)}(z_0) \neq 0$.

Conversely, let $W(z_0) = W'(z_0) = \cdots = W^{(n-1)}(z_0) = 0$ and $W^{(n)}(z_0) \neq 0$. …(2.1)

Then clearly z_0 is a zero of $W(z)$, say of order l.

Thus there exists a positive integer t such that $\left[\left|\frac{t}{k}\right|\right] = l$ and $W(z) = (z - z_0)^{\frac{t}{k}} W_0(z)$,

where $W_0(z_0) \neq 0$ and by first part

$W(z_0) = W'(z_0) = \cdots = W^{(l-1)}(z_0) = 0$ and $W^{(l)}(z_0) \neq 0$. (2.2)

From (2.1) and (2.2), we have $l = n$

This completes the proof of Theorem 2.1.

Theorem 2.2: Let $W(z)$ be a k-valued entire algebroidal function having only finitely many zeros. Then $W(z) = [P(z)]^{\frac{1}{k}} e^{\psi(z)}$, for some polynomial $P(z)$ and an entire function $\psi(z)$.

Proof: Let $z_1, z_2, \ldots, z_p$ be the zeros of $W(z)$ of order $n_1, n_2, \ldots, n_p$, respectively. Then there exists positive integers $m_1, m_2, \ldots,, m_p$ such that $n_i = \left[\left|\frac{m_i}{k}\right|\right]$, $i = 1, 2, .., p$ and

$$W(z) = (z - z_1)^{\frac{m_1}{k}} (z - z_2)^{\frac{m_2}{k}} \cdots (z - z_p)^{\frac{m_p}{k}} \,\square(z),$$

where □(z) is an entire function having no zeros. Thus,

$$W(z) = \left[(\mathbf{z} - \mathbf{z_1})^{m_1} (\mathbf{z} - \mathbf{z_2})^{m_2} \ldots (\mathbf{z} - \mathbf{z_p})^{m_p}\right]^{\frac{1}{k}} \,\square(z)$$

$$= [P(z)]^{\frac{1}{k}} \,\square(z),$$

where $P(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_p)^{m_p}$ is a polynomial. Since □(z) is an entire function having no-zeros, so □(z) = $e^{\square(z)}$, where □(z) is an entire function. This completes the proof of Theorem 2.2.

<u>Remark 2.1:</u> Let W(z) be an entire algebroidal function of finite order. If W(z) has finite number of zeros, then $W'(z)$ has also finite number of zeros. This follows from Theorem 2.2. In this case □(z) is a polynomial. But if W(z) is an entire algebroidal function of infinite order, then the above statement is not true. For, consider

$$W(z) = z^{\frac{3}{2}} e^{e^z}$$

Then W(z) is an entire algebroidal function of infinite order having finite number of zeros but $W'(z)$ has infinite number of zeros.

<u>Theorem 2.3:</u> Let □(z) be an entire function and let p and k be positive integers such that p < k. Then $W(z) = z^{\frac{p}{k}} e^{\square(z)}$ is prime in divisor sense.

<u>Proof:</u> Let

$$W(z) = W_0(g(z))\, e^{\square(z)},$$

where $W_0(z)$ is k_0-valued entire algebroidal function, $k_0 \le k$, g(z) ($\not\equiv$ constant) and □(z) are entire. Then $W_0(g(z))$ has just one zero. Thus, there exists z_0 such that $W_0(g(z_0)) = 0$ and $W_0(z_1) = 0$ for unique $z_1 = g(z_0)$. Therefore, z_1 is a zero of W_0.

<u>Case (i):</u> Suppose g(z) is transcendental. Let z_2 ($\neq z_1$) be a zero of W_0. Then $g(z) = z_i$. for some fixed i = 1, 2, has infinity of solutions say $\{w_j\}_{j=1}^{\infty}$. Thus, $W_0(g(z))$ has infinity of zeros, a contradiction.

Suppose z_1 is a zero of order n (≥ 2) of W_0.

Then there exists a positive integer m such that $n = \left[\left|\frac{m}{k_0}\right|\right]$ and $W(z) = (z - z_1)^{\frac{m}{k_0}} \,\square(z)$,

where □(z) has no-zeros. Thus,

$$W_0(g(z)) = (g(z) - z_1)^{\frac{m}{k_0}} \,\square(g(z))$$

$$= (g(z) - z_1)^{\frac{m}{k_0}} \psi(z),$$

where □(z) has no-zeros.

<u>Subcase(i):</u> When $g(z) = z_1$ has no solution. Then $W_0(g(z))$ has no-zeros, a contradiction.

<u>Subcase(ii):</u> When $g(z) = z_1$ has a solution. Then $W_0(g(z))$ has a zero of order $n \ge 2$, again a contradiction. Thus, if g is transcendental, then W_0 has just one simple zero.

Case (ii): Suppose g(z) is a polynomial of degree $\geq$ 2. Then g(z) – z_1 has at least two zeros.

Subcase (i): Suppose g(z) – z_1 has two distinct zeros, say, $\square_1$ and $\square_2$. Then g($\square_1$) = z_1 and g($\square_2$) = z_1, so W_0(g($\square_1$)) = 0 and W_0(g($\square_2$)) = 0.
Thus, W_0 (g(z)) has two distinct zeros, a contradiction.

Subcase (ii): Suppose g(z) – z_1 has a zero of order n ($\geq$ 2) at $\square_1$, say. Then W_0(g($\square_1$)) = 0 and $[W_0(g)]'$ ($\square_1$) = $W_0'(g(\square_1))\, g'(\square_1)$

= 0

Thus, W_0(g(z)) has a zero of order n $\geq$ 2, again a contradiction. Hence case (ii) is not possible.

Case (iii): Suppose g is linear polynomial. In this case there is nothing to prove.
Hence W(z) is prime in divisor sense.

Remark 2.2: If p > k, then the above theorem may not be true. For, consider W(z) = $z^{\frac{p}{k}}$.

Let $W_0(z) = z^{\frac{p}{k}}$ and $g(z) = z\,e^{B(z)}$, where B(z) is non-constant entire function.

$$\text{Then } W_0(g(z))\; e^{-\frac{p}{k}B(z)} = W_0(z\,e^{B(z)})\; e^{-\frac{p}{k}B(z)}$$

$$= (z\,e^{B(z)})^{\frac{p}{k}}\; e^{-\frac{p}{k}B(z)}$$

$$= W(z)$$

Thus, W(z) is not prime in divisor sense.

Remark 2.3: An algebroidal function which is prime may not be prime in divisor sense.

For, consider W(z) = $z^{\frac{p}{n}}$, where p is prime > n. Then W(z) satisfies $W^n - z^p = 0$ and by Definition 1.4, W(z) is prime. Clearly W(z) is not prime in divisor sense in view of
Remark 2.2.

Remark 2.4: An algebroidal function which is prime in divisor sense may not be prime.

For, consider W(z) = $z^{\frac{1}{2}}\, e^{z\left(e^z+\frac{1}{2}\right)}$. Then by Theorem 2.2, W(z) is prime in divisor sense. But W(z) is not prime. For, let

$W_0(z) = z^{\frac{1}{2}}\, e^z$ and $g(z) = z\,e^z$. Then

$W_0(g(z)) = W_0(z\,e^z)$

$= W(z)$

REFERENCES

1. Zheng, J.H. : The growth and factorization of composition of algebroidal functions, Natural Magazine, 12:11, 1989, pp. 876.
2. Zheng, J.H. and Yang, C.C. : Facotrization of algebroidal functions, Chinese Journal of Contemporary Mathematics Vol. 13, No.4(1992), 333-340.
3. Urabe, H: Primeness in divisor sense of certain entire functions. J. Math. Kyoto Univ. 25(1), 1984, 127-140.

4. Urabe H: On pseudo-primeness in divisor sense for entire functions and its application to the factorization of meromorphic functions. Bull. Kyoto. Univ. of Education Ser. B. 81(1992), 1-16.
5. Singh A.P. and Sharma, J.L: Some criteria for determining pseudo-primeness in divisor sense of entire functions. Proc. Nat. Acad. Sci. India, 68(A) IV, 1998, 365-372
6. Singh, A.P. and Sharma, J.L.: On factorization of meromorphic functions. Bull. Allahabad Math Soc. Vol. 15, 2000, 9-15.