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Asymptotic Properties of Impulsive Neutral Nonlinear Partial Differential Equations

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Abstract

In this paper, we consider the asymptotic behavior of solutions of impulsive neutral nonlinear partial differential equations. Some new sufficient conditions are obtained for every solution of the equation that tends to a constant as $t \to \infty$ by using Riccati transformation and impulsive differential inequality technique. Our results extend and improve some of the related results reported in the literature. An example is given to illustrate the effectiveness of our results.

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1 Introduction

The theory of impulsive differential equations is now being recognized as not only richer than the corresponding theory of differential equations without impulses but also representing a more natural framework for mathematical modelling of many real-world phenomena. The theory of impulsive differential equations has been less developed due to numerous theoretical and technical difficulties caused by their peculiarities. Also, theory of impulsive differential equations arises in modeling dynamical systems with discontinuous trajectories. These problems are emerging in nonlinear mechanics dealing with the process in nonlinear oscillating system. The literature on applications of qualitative theory of impulsive differential equations are growing rapidly day by day. This study is a relatively new field and is very interesting in applications in some mathematical models in many biological phenomena involving thresholds, bursting rhythm model (medicine and biology), ecology, epidemiology, disease modeling, optimal control model (economics), industrial robotics, biotechnology and spread of some infectious diseases (humans), etc., can be expressed by impulsive delay differential equations. These processes and phenomena, which adequate mathematical models are impulsive delay differential equations, are characterized by the fact per sudden changing of their state and that the processes under consideration depend on their prehistory at each moment of time.

Recently neutral delay differential equations, that is, equations in which the highest order derivative of the unknown function appears both with and without delays, have received strong interest in the study of oscillation properties of their solutions. The problem of asymptotic and oscillatory behavior of solutions of neutral equations is of both theoretical and practical interest. One reason for this is that they arise for example, in applications to electric networks containing loss-less transmission lines such networks appear in high speed computers where lossless transmission lines are used to interconnect switching circuits. They also occur in problems dealing with vibrating masses attached to an elastic bar and in the solution of variational problems with time delays. Interested readers can refer to the books $[1, 3, 12, 13, 20, 22]$ for some applications in science and technology.

In the last few decades, the oscillation and asymptotic behavior of solutions of differential equations with impulses have been extensively studied by a number of researchers [2,4–6,9–11,14,15,18,19,23] and the references quoted therein.

In [7], Jiao et.al studied the asymptotic behavior of solutions of second order nonlinear impulsive differential equations of the form

$$\begin{aligned} &(r(t)x'(t))' + p(t)x' + Q(t, x(t)) = 0, \quad t \neq t_k, \\ &x'(t_k^+) = M_k(x'(t_k)), \quad x(t_k^+) = N_k(x(t_k)), \quad k = 1, 2, \cdots, \quad t \geq t_0. \end{aligned}$$

In [16], Tang studied the asymptotic behavior of the following differential equations with impulses of the form

$$\begin{aligned} &(r(t)x'(t))' + p(t)x'(t) + f(t, x(t-\delta)) = 0, \quad t \neq t_k \\ &x'(t_k^+) = I_k x'(t_k), \quad x(t_k^+) = J_k x(t_k), \quad k = 1, 2, \cdots. \end{aligned}$$

To the best our knowledge, it seems that there has been no paper dealing with asymptotic behavior of impulsive neutral partial differential equations. Motivated by this gap, we propose to initiate the following model of the form

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left[r(t)g\left(\frac{\partial}{\partial t}\left(u(x,t)+h(t)u(x,\rho(t))\right)\right)\right] + p(t)g\left(\frac{\partial}{\partial t}\left(u(x,t)+h(t)u(x,\rho(t))\right)\right) \\ &+q(x,t)f(u(x,\sigma(t))) + \sum_{i=1}^{n} q_i(x,t)f_i(u(x,\sigma(t))) \\ &= a(t)\Delta u(x,t) + \sum_{j=1}^{m} a_j(t)\Delta u(x,\delta_j(t)) + E(x,t), \quad t \neq t_k, \\ &u(x,t_k^+) = a_k(x,t_k,u(x,t_k)) \\ &u_t(x,t_k^+) = b_k(x,t_k,u_t(x,t_k)), \quad k = 1,2,\cdots, \qquad (x,t) \in \Omega \times \mathbb{R}_+ \equiv G, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+ = [0, +\infty)$.

Equation (1.1) is enhancement with one of the subsequent Robin boundary condition,

$$\frac{\partial}{\partial \gamma}u(x,t) + \mu(x,t)u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{1.2}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t) \in C\left(\partial\Omega \times [0,+\infty), [0,+\infty)\right)$.

In this paper, we assume that the following hypotheses (H) hold:

(H_1) $r(t) \in C^1(\mathbb{R}_+, (0,+\infty))$, $p(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\int_{t_0}^{\infty} \frac{1}{R(s)}ds = \infty$, where $R(t) = exp\left(\int_{t_0}^{t} \frac{r'(s)+p(s)}{r(s)}ds\right)$ $q(x,t), q_i(x,t) \in C(\bar{G}, \mathbb{R}_+)$, $q(t) = \min_{x\in\bar{\Omega}} q(x,t)$, $q_i(t) = \min_{x\in\bar{\Omega}} q_i(x,t)$, $i = 1,2,\cdots,n$, $f, f_i \in C(\mathbb{R},\mathbb{R})$ are convex in $\mathbb{R}_+$ with $uf(u) > 0$, $uf_i(u) > 0$ and $\frac{f(u)}{u} \geq \epsilon > 0$, $\frac{f_i(u)}{u} \geq \epsilon_i > 0$ for $u \neq 0$, $i = 1,2,\cdots,n$, $\rho(t), \sigma(t), \delta_j(t) \in C(\mathbb{R}_+,\mathbb{R})$, $\lim_{t\to+\infty} \rho(t) = \lim_{t\to+\infty} \sigma(t) = \lim_{t\to+\infty} \delta_j(t) = +\infty$, $j = 1,2,\cdots,m$ and $h(t) \in C^2(\mathbb{R}_+,\mathbb{R}_+)$.

(H_2) $E \in C(\bar{G},\mathbb{R}), g \in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+$ with $ug(u) > 0$, $g(u) \leq cu$, for $u \neq 0$, $g^{-1} \in C(\mathbb{R},\mathbb{R})$ is continuous function with $ug^{-1}(u) > 0$ for $u \neq 0$ and there exist positive constant θ such that $g^{-1}(uv) \leq \theta g^{-1}(u)g^{-1}(v)$ for $uv \neq 0$ and $\int_\Omega E(x,t)dx \leq 0$.

(H_3) $a(t)$, $a_j(t) \in PC\left(\mathbb{R}_+,\mathbb{R}_+\right)$, $j = 1,2,\cdots,m$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1,2,\cdots$, and left continuous at $t = t_k$, $k = 1,2,\cdots$.

(H_4) $u(x,t)$ and its derivative $u_t(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1,2,\cdots$, and left continuous at $t = t_k$, $u(x,t_k) = u(x,t_k^-)$, $u_t(x,t_k) = u_t(x,t_k^-)$, $k = 1,2,\cdots$.

(H_5) $a_k, b_k \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $k = 1, 2, \cdots$, and there exist positive constants α_k, α_k^*, β_k, β_k^* such that $\alpha_k^* \leq \alpha_k \leq \beta_k^* \leq \beta_k$ for $k = 1, 2, \cdots$,

$$\alpha_k^* \leq \frac{a_k\left(x, t_k, u(x, t_k)\right)}{u(x, t_k)} \leq \alpha_k, \quad \beta_k^* \leq \frac{b_k\left(x, t_k, u_t(x, t_k)\right)}{u_t(x, t_k)} \leq \beta_k.$$

(H_6) $$\lim_{t \to +\infty} \int_{t_s}^{t} \prod_{s < t_k < t} \frac{\beta_k^*}{\alpha_k} \exp\left(-\int_s^t \frac{r'(\xi) + p(\xi)}{r(\xi)} d\xi\right) ds = +\infty.$$

(H_7) $$\sum_{m=1}^{n-1} \prod_{k=m}^{n-1} \prod_{l=0}^{m-1} \beta_k \alpha_l^* \int_{t_{m-1}}^{t_m} \exp\left(\int_{t_0}^{u} \frac{r'(\xi) + p(\xi)}{r(\xi)} d\xi\right) du + \prod_{k=0}^{n-1} \alpha_k \int_{t_{n-1}}^{t_n} \exp\left(-\int_{t_0}^{u} \frac{r'(\xi) + p(\xi)}{r(\xi)} d\xi\right) du = +\infty.$$

Definition 1. *A solution u of the problem (1.1)-(1.2) is a function $u \in C^2(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (1.1), where*

$$t_{-1} := \min\left\{0, \min_{1 \leq j \leq m}\left\{\inf_{t \geq 0} \delta_j(t)\right\}, \left\{\inf_{t \geq 0} \rho(t)\right\}\right\}, \quad \hat{t}_{-1} := \min\left\{0, \inf_{t \geq 0} \sigma(t)\right\}.$$

Definition 2. *The solution u of the problem (1.1)-(1.2) is said to be eventually positive (negative) if it is positive (negative) for all sufficiently large t. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is non-oscillatory.*

For convenience, we introduce the following notations:

$$\Gamma_k = \{(x, t) : \ t \in (t_k, t_{k+1}], x \in \Omega\}, \quad \Gamma = \bigcup_{k=0}^{\infty} \Gamma_k,$$
$$\overline{\Gamma}_k = \{(x, t) : \ t \in (t_k, t_{k+1}], x \in \overline{\Omega}\}, \quad \overline{\Gamma} = \bigcup_{k=0}^{\infty} \overline{\Gamma}_k,$$
$$v(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx, \quad Q(t) = \epsilon q(t) + \sum_{j=1}^{m} \epsilon_j q_j(t),$$
$$\text{where} |\Omega| = \int_{\Omega} dx, \quad h_0 = 1 - h(\sigma(t)).$$

This paper is organized as follows: In Section 2, basic lemmes which are useful in sequel and also we obtain some new sufficient conditions of asymptotic behavior of the impulsive neutral differential equation (1.1), together with the boundary condition (1.2). In Section 3, an example is presented to illustrate the main result.

2 Main Results

The following lemmas useful for the main results.

Lemma 2.1. *Let $u \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1.1)-(1.2) in G. Then the function $z(t)$ satisfies the following impulsive differential inequality*

$$\left.\begin{aligned} &(r(t) g(z'(t)))' + p(t) g(z'(t)) + c_0 Q(t) z(\sigma(t)) \leq 0, \quad t \neq t_k \\ &\alpha_k^* \leq \frac{z(t_k^+)}{z(t_k)} \leq \alpha_k, \quad \beta_k^* \leq \frac{z'(t_k^+)}{z'(t_k)} \leq \beta_k, \quad k = 1, 2, \cdots. \end{aligned}\right\} \tag{2.1}$$

Proof. Let u be a positive solution of the problem $(1.1) - (1.2)$ in G. Without loss of generality, we may assume that $u(x, t) > 0$, $u(x, \rho(t)) > 0$, $u(x, \sigma(t)) > 0$ and $u(x, \delta_j(t)) > 0$, $j = 1, 2, \cdots, m$ for any $(x, t) \in \Omega \times [t_0, +\infty)$.

For $t \geq t_0,\ t \neq t_k,\ k = 1, 2, \cdots$, multiplying both sides of equation (1.1) by $\dfrac{1}{|\Omega|}$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned}
&\frac{d}{dt}\left[r(t)g\left(\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx + h(t)\frac{1}{|\Omega|}\int_\Omega u(x,\rho(t))dx\right)\right)\right]\\
&+p(t)g\left(\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx + h(t)\frac{1}{|\Omega|}\int_\Omega u(x,\rho(t))dx\right)\right)\\
&+\frac{1}{|\Omega|}\int_\Omega q(x,t)f(u(x,\sigma(t)))dx + \sum_{i=1}^{n}\frac{1}{|\Omega|}\int_\Omega q_i(x,t)f_i(u(x,\sigma(t)))dx\\
&= a(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,t)dx + \sum_{j=1}^{m} a_j(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,\delta_j(t))dx + \frac{1}{|\Omega|}\int_\Omega E(x,t)dx.
\end{aligned}\right\} \tag{2.2}$$

From Green's formula and boundary condition (1.2), we see that

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega}\frac{\partial u}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,t)u(x,t)dS \leq 0 \tag{2.3}$$

and for $j = 1, 2, \cdots, m$, we have

$$\int_\Omega \Delta u(x,\delta_j(t))dx = \int_{\partial\Omega}\frac{\partial u(x,\delta_j(t))}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,\delta_j(t))u(x,\delta_j(t))dS \leq 0 \tag{2.4}$$

where dS is surface element on $\partial\Omega$. Moreover using Jensen's inequality, from (H_1) and assumptions, it follows that

$$\frac{1}{|\Omega|}\int_\Omega q(x,t)f(u(x,\sigma(t)))dx \geq \epsilon q(t)\frac{1}{|\Omega|}\int_\Omega u(x,\sigma(t))dx \geq \ \epsilon q(t)v(\sigma(t)), \tag{2.5}$$

and for $i = 1, 2, \cdots, n$

$$\frac{1}{|\Omega|}\int_\Omega q_i(x,t)f_i(u(x,\sigma(t)))dx \geq \epsilon_i q_i(t)\frac{1}{|\Omega|}\int_\Omega u(x,\sigma(t))dx \geq \ \epsilon_i q_i(t)v(\sigma(t)). \tag{2.6}$$

In view of (2.2)-(2.6), we obtain

$$\frac{d}{dt}\left[r(t)g\left(\frac{d}{dt}\left(v(t)+h(t)v(\rho(t))\right)\right)\right] + p(t)g\left(\frac{d}{dt}\left(v(t)+h(t)v(\rho(t))\right)\right) + \epsilon q(t)v(\sigma(t)) + \epsilon_i q_i(t)v(\sigma(t)) \leq 0.$$

Set $z(t) = v(t) + h(t)v(\rho(t))$. Then

$$(r(t)g(z'(t)))' + p(t)g(z'(t)) + Q(t)v(\sigma(t)) \leq 0, \qquad t \neq t_k. \tag{2.7}$$

It is easy to obtain that $z(t) > 0$ for $t \geq t_1$. Next we prove that $g(z'(t)) > 0$ for $t \geq t_2$. Assume the contrary, there exists $T \geq t_2$ such that $g(z'(T)) \leq 0$.

$$\begin{aligned}
(r(t)g(z'(t)))' + p(t)g(z'(t)) &\leq 0, \qquad t \geq t_2\\
r(t)g'(z'(t))z''(t) + (r'(t)+p(t))\,g(z'(t)) &\leq 0, \qquad t \geq t_2.
\end{aligned} \tag{2.8}$$

From (H_1), we have $R'(t) = R(t)\left(\dfrac{r'(t)+p(t)}{r(t)}\right)$ and $R(t) > 0,\quad R'(t) \geq 0$ for $t \geq t_2$. We multiply $\dfrac{R(t)}{r(t)}$ on both sides of (2.8), we have

$$R(t)g(z'(t))z''(t) + R'(t)g(z'(t)) = (R(t)g(z'(t)))' \leq 0, \quad t \geq t_2. \tag{2.9}$$

From (2.9), we have $R(t)g(z'(t)) \leq R(T)g(z'(T)) \leq 0,\ t \geq T$. Thus

$$\int_T^t g(z'(s))ds \leq \int_T^t \frac{cR(T)z'(T)}{R(s)}ds, \quad t \geq T$$

$$z(t) \leq \ z(T) + R(T)z'(T)\int_T^t \frac{ds}{R(s)}, \quad t \geq T.$$

From the hypotheses (H_1), we have $\lim_{t\to+\infty} z(t) = -\infty$. This contradicts that $z(t) > 0$ for $t \geq 0$. Thus $z'(t) > 0$, $\sigma(t) \leq t$ and $\rho(t) \leq t$ for $t \geq t_1$, we have

$$\begin{aligned} v(t) =&\ z(t) - h(t)v(\rho(t)) \\ v(t) \geq&\ z(t) - h(t)z(t) \\ v(t) \geq&\ z(t)(1 - h(t)) \end{aligned}$$

and

$$\begin{aligned} v(\sigma(t)) \geq&\ z(\sigma(t))(1 - h(\sigma(t)) \\ v(\sigma(t)) \geq&\ h_0 z(\sigma(t)). \end{aligned}$$

Therefore from (2.7), we have

$$(r(t)g(z'(t)))' + p(t)g(z'(t)) + h_0 Q(t)z(\sigma(t)) \leq 0, \qquad t \geq t_1.$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \cdots$, multiplying both sides of the equation (1.1) by $\frac{1}{|\Omega|}$, integrating with respect to x over the domain Ω, and from (H_4), we obtain

$$\alpha_k^* \leq \frac{u(x,t_k^+)}{u(x,t_k)} \leq \alpha_k, \qquad \beta_k^* \leq \frac{u_t(x,t_k^+)}{u_t(x,t_k)} \leq \beta_k.$$

From assumptions we have,

$$\alpha_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq \alpha_k, \qquad \beta_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq \beta_k,$$

and

$$\alpha_k^* \leq \frac{z(t_k^+)}{z(t_k)} \leq \alpha_k, \qquad \beta_k^* \leq \frac{z'(t_k^+)}{z'(t_k)} \leq \beta_k.$$

Hence we obtain that $z(t)$ is a solution of impulsive inequality (2.1). This completes the proof. □

Lemma 2.2. *[17] Assume that conditions* $(H_1) - (H_5)$*hold and let* $u(x,t)$ *be a positive solution of* (1.1)-(1.2). *Then for sufficiently large* t*, either*

(i) $z(t) > 0, \quad z'(t) > 0, \quad (r(t)g(z'(t)))' < 0,$ *(or)*

(ii) $z(t) > 0, \quad z'(t) < 0, \quad (r(t)g(z'(t)))' < 0.$

Lemma 2.3. *Assume that conditions* $(H_1) - (H_5)$ *hold and let* $u(x,t)$ *be an eventually positive solution of* (1.1)-(1.2) *with* $z(t)$ *satisfying case* (ii). *If*

$$\frac{1}{c}\int_{t_1}^{\infty}\int_{t}^{\infty} \frac{1}{r(u)}\left[p(s)z'(s) + Q(s)Z(\theta(s))\right] ds du = \infty. \tag{2.10}$$

then $\lim_{t\to\infty} z(t) = 0$.

Proof. Let $u(x,t)$ be an eventually positive solution of (1.1)-(1.2). Then $z(t)$ satisfies the inequality (2.1).

$$(r(t)g(z'(t))' \leq -p(t)g(z'(t)) - h_0 Q(t)z(\sigma(t)) \leq 0.$$

By Lemma 2.2, there exist a constant l such that $\lim_{t\to\infty} z(t) = l < \infty$. Integrating the above inequality from t to ∞, we get

$$\begin{aligned} r(t)g(z'(t)) \geq& \int_t^{\infty} (p(s)g(z'(s)) + h_0 Q(s)z(\sigma(s)))\, ds, \\ g(z'(t)) \geq& \int_t^{\infty} (p(s)g(z'(s)) + h_0 Q(s)z(\sigma(s)))\, ds, \\ z'(t) \geq& \frac{1}{cr(t)}\int_t^{\infty} (p(s)g(z'(s)) + h_0 Q(s)z(\sigma(s)))\, ds. \end{aligned}$$

Again integrating from t_1 to ∞, we obtain

$$z(t) \leq -\frac{1}{c}\int_{t_1}^{\infty}\frac{1}{r(u)}\int_{t}^{\infty}\left(p(s)g(z'(s)) + h_0 Q(s)z(\sigma(s))\right)dsdu,$$

we have a contradiction with (2.10), and so we have $l = 0$. Therefore $\lim_{t\to\infty} z(t) = 0$.
This complete the proof . □

Lemma 2.4. *[8] Assume that*

(A1) the sequence $\{t_k\}$ *satisfies* $0 < t_0 < t_1 < \cdots,\ \lim_{k\to\infty} t_k = \infty$*;*

(A2) $m(t) \in PC^1[\mathbb{R}^+, \mathbb{R}]$ *is left continuous at* t_k *for* $k = 1, 2, \cdots$ *;*

(A3) for $k = 1, 2, \cdots$ *and* $t \geq t_0$*,*

$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k,$$
$$m(t_k^+) \leq d_k m(t_k) + e_k,$$

where $p(t),\ q(t) \in C(\mathbb{R}^+, \mathbb{R})$*,* $d_k \geq 0$ *and* e_k *are constants. PC denote the class of piecewise continuous function from* $\mathbb{R}^+$ *to* $\mathbb{R}$*, with discontinuities of the first kind only at* $t = t_k$*,* $k = 1, 2, \cdots$*.*

Then

$$\begin{aligned} m(t) \ &\leq m(t_0)\prod_{t_0<t_k<t} d_k\, exp\left(\int_{t_0}^{t} p(s)ds\right) + \int_{t_0}^{t}\prod_{s<t_k<t} d_k\, exp\left(\int_{s}^{t} p(r)dr\right) q(s)ds \\ &+ \sum_{t_0<t_k<t}\prod_{t_k<t_j<t} d_j\, exp\left(\int_{t_k}^{t} p(s)ds\right) e_k. \end{aligned}$$

Lemma 2.5. *Assume that the conditions of Lemma 2.1 holds. Suppose that there exist some* $T \geq t_0$ *such that* $z(t) > 0$*,* $t \geq T$*. If* $(H_1) - (H_6)$ *are satisfied then* $g(z'(t_k)) > 0$ *and* $g(z'(t)) > 0$ *for* $t \in (t_k, t_{k+1}]$ *where* $t_k \geq T$*,* $k = 1, 2, \cdots$*.*

Proof. Let $z(t) > 0$, $t \geq T$, we will prove that $g(z'(t_k)) > 0$, for any $t_k \geq T$, $T \geq t_0$. If not, there must exist some ℓ such that $g(z'(t_\ell)) < 0$, $t_\ell \geq T$ and $g(z'(t_\ell^+)) \leq \beta_\ell^* g(z'(t_\ell)) < 0$. Then, let

$$g(z'(t_\ell))\exp\left(\int_{t_0}^{t_\ell}\frac{r'(s)+p(s)}{r(s)}ds\right) = \Theta < 0.$$

From (2.1), it is clear that

$$\left(g(z'(t))\exp\left(\int_{t_0}^{t_\ell}\frac{r'(s)+p(s)}{r(s)}ds\right)\right)' \leq -h_0\frac{Q(t)}{r(t)}z(\sigma(t))\exp\left(\int_{t_0}^{t_\ell}\frac{r'(s)+p(s)}{r(s)}ds\right) < 0. \tag{2.11}$$

Hence, the function $g(z'(t))\exp\left(\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)$ is nonincreasing on $(t_\ell, t_{\ell+1}]$,

$$g(z'(t_{\ell+1}))\exp\left(\int_{t_0}^{t_{\ell+1}}\frac{r'(s)+p(s)}{r(s)}ds\right) < g(z'(t_\ell))\exp\left(\int_{t_0}^{t_\ell^+}\frac{r'(s)+p(s)}{r(s)}ds\right),$$

that is,

$$g(z'(t_{\ell+1}))\exp\left(\int_{t_0}^{t_{\ell+1}}\frac{r'(s)+p(s)}{r(s)}ds\right) \leq \beta_\ell^* g(z'(t_\ell^+))\exp\left(\int_{t_0}^{t_\ell^+}\frac{r'(s)+p(s)}{r(s)}ds\right) \leq \beta_\ell^*\Theta$$

and

$$g(z'(t_{\ell+2}))\exp\left(\int_{t_0}^{t_{\ell+2}}\frac{r'(s)+p(s)}{r(s)}ds\right)\le g(z'(t_{\ell+1}^+))\exp\left(\int_{t_0}^{t_{\ell+1}^+}\frac{r'(s)+p(s)}{r(s)}ds\right),$$

and

$$g(z'(t_{\ell+2}))\exp\left(\int_{t_0}^{t_{\ell+2}}\frac{r'(s)+p(s)}{r(s)}ds\right)\le \beta_{\ell+1}^*\beta_\ell^*\Theta.$$

By induction, we obtain

$$g(z'(t_{\ell+n}))\exp\left(\int_{t_0}^{t_{\ell+n}}\frac{r'(s)+p(s)}{r(s)}ds\right)\le \prod_{k=0}^{n-1}\beta_{\ell+k}^*\Theta,$$

while for $t\in(t_{\ell+n},t_{\ell+n+1}]$, we derive

$$g(z'(t))\exp\left(\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)\le g(z'(t_{\ell+n}^+))\exp\left(\int_{t_0}^{t_{\ell+n}^+}\frac{r'(s)+p(s)}{r(s)}ds\right)\le \prod_{t_\ell<t_k<t}\beta_k^*\Theta.$$

Then, we obtain

$$g(z'(t))\le \prod_{t_\ell<t_k<t}\beta_k^*\Theta\exp\left(-\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right). \tag{2.12}$$

From the condition $z(t_n^+)\le\alpha_n z(t_n)$, we have the following impulsive differential inequality

$$\left.\begin{array}{l} z'(t)\le \frac{1}{c}\prod_{t_\ell<t_k<t}\beta_k^*\Theta\exp\left(-\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right),\quad t\ne t_k,\quad k=\ell+1,\ell+2,\cdots\\ z(t_k^+)\le\alpha_k z(t_k),\quad t\ge t_\ell. \end{array}\right\} \tag{2.13}$$

Applying Lemma 2.4, we obtain

$$z(t)\le z(t_\ell^+)\prod_{t_\ell<t_k<t}\alpha_k+\frac{\Theta}{c}\int_{t_\ell}^{t}\prod_{s<t_k<t}\alpha_k\prod_{t_\ell<t_i<s}\beta_i^*\exp\left(-\int_{t_\ell}^{s}\frac{r'(\xi)+p(\xi)}{r(\xi)}d\xi\right)ds. \tag{2.14}$$

In view of the fact that

$$\prod_{t_\ell<t_k<t}\alpha_k=\prod_{t_\ell<t_i<s}\alpha_i\prod_{s<t_\ell<t}\alpha_l.$$

So we see that

$$z(t)\le\prod_{t_\ell<t_k<t}\alpha_k\left\{z(t_\ell^+)+\frac{\Theta}{c}\int_{t_\ell}^{t}\prod_{t_\ell<t_i<s}\frac{\beta_i^*}{\alpha_i}\exp\left(-\int_{s}^{t}\frac{r'(\xi)+p(\xi)}{r(\xi)}d\xi\right)ds\right\}. \tag{2.15}$$

Since $z(t_k)>0$ $(t_k\ge T)$, one can find that (2.15) contradicts condition (H_6) as $t\to+\infty$, therefore $g(z'(t_k))\ge 0$ $(t\ge T)$. By (2.1), we have $g(z'(t_k^+))\ge\beta_k^* g(z'(t_k))$ for any $t_k\ge T$. Because $g(z'(t))\exp\left(\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)$ is nonincreasing on $(t_{\ell+i-1},t_{\ell+i}]$, we have $g(z'(t))\exp\left(\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)>0$ for any $t\in(t_{\ell+i-1},t_{\ell+i}]$, which implies $g(z'(t))\ge 0$ $(t\ge T)$. This complete the proof. □

Lemma 2.6. *[21] Assume that the conditions of Lemma 2.1 holds. Suppose that there exist some $T\ge t_0$ such that $z(t)>0$, $t\ge T$. If $(H_1)-(H_5)$ and (H_7) are satisfied, then $g(z'(t_\ell))>0$ and $g(z'(t))>0$ for $t\in(t_k,t_{k+1}]$, where $t_k\ge T$, $k=1,2,\cdots$.*

Next, we have Riccati transformation, we firstly transform (2.1) into a Riccati equation. Then we investigate the asymptotic behavior of all solutions of neutral nonlinear partial differential equations with impulses by impulsive differential inequality, and obtain the following two theorems.

Theorem 2.1. *If the conditions* $(H_1) - (H_6)$ *and*

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{\alpha_k^*}{\beta_k}\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)h_0Q(s)ds=+\infty \tag{2.16}$$

are satisfied, then every solution $u(x,t)$ *of* (1.1)-(1.2) *satisfies* $\liminf_{t\to\infty}|u(x,t)|=0$.

Proof. Let $u(x,t)$ be a solution of (1.1)-(1.2) and suppose by contradiction that $\liminf_{t\to\infty}|u(x,t)|>0$. So $u(x,t)$ is non-oscillatory. Without loss of generality, we may assume that $u(x,t)>0$ on $(T_0,+\infty)$ for some large $T_0\geq t_0$. By Lemma 2.5, $z'(t)>0$ for all $t\geq T_0$. We consider a Riccati transformation as the following form:

$$w(t):=\frac{r(t)g(z'(t))}{z(\sigma(t))}. \tag{2.17}$$

Differentiating $w(t)$, we have

$$\begin{aligned}w'(t)&=\frac{(r(t)g(z'(t)))'z(\sigma(t))-r(t)g(z'(t))z'(\sigma(t))\sigma'(t)}{z^2(\sigma(t))}\\&\leq-\frac{p(t)}{r(t)}w(t)-h_0Q(t)-\frac{w^2(t)}{r(t)}\sigma'(t)\\&\leq-\frac{p(t)}{r(t)}w(t)-h_0Q(t).\end{aligned} \tag{2.18}$$

From (2.17) and (2.1), it is clear that

$$w(t_k^+)=\frac{r(t_k^+)g(z'(t_k^+))}{z(\sigma(t_k^+))}\leq\frac{\beta_k}{\alpha_k^*}w(t_k). \tag{2.19}$$

Integrating (2.18) with (2.19), we have

$$\left.\begin{aligned}w'(t)&\leq-\frac{p(t)}{r(t)}w(t)-h_0Q(t),\quad t\neq t_k\\w(t_k^+)&\leq\frac{\beta_k}{\alpha_k^*}w(t_k),\quad k=1,2,\cdots.\end{aligned}\right\} \tag{2.20}$$

Applying Lemma 2.4, we get

$$w(t)\leq w(t_0)\prod_{t_0<t_k<t}\frac{\beta_k}{\alpha_k^*}\exp\left(-\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)-h_0\int_{t_0}^{t}\prod_{s<t_k<t}\frac{\beta_k}{\alpha_k^*}\exp\left(-\int_{s}^{t}\frac{p(\xi)}{r(\xi)}d\xi\right)Q(s)ds.$$

So we have

$$w(t)\leq\prod_{t_0<t_k<t}\frac{\beta_k}{\alpha_k^*}\exp\left(-\int_{t_0}^{t}\frac{p(\xi)}{r(\xi)}d\xi\right)\left\{w(t_0)-h_0\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{\alpha_k^*}{\beta_k}\exp\left(\int_{t_0}^{s}\frac{p(\xi)}{r(\xi)}d\xi\right)Q(s)ds\right\}.$$

By the condition (2.16), the above inequality is impossible. Then by Lemma 2.3, we get $\lim_{t\to\infty}z(t)=0$. Since $0<v(t)<z(t)$ on $[t_1,\infty)$, we obtain $\lim_{t\to\infty}v(t)=0$. This implies

$$\lim_{t\to\infty}\frac{1}{|\Omega|}\int_{\Omega}u(x,t)dx=0,$$
$$\lim_{t\to\infty}|u(x,t)|=0.$$

This proof is complete. ∎

Theorem 2.2. *If the conditions* $(H_1)-(H_6)$, (2.16) *and* (H_7) *are satisfied, then every solution* $u(x,t)$ *of* (1.1)-(1.2) *satisfies* $\liminf_{t\to\infty}|u(x,t)|=0$.

Proof. The proof is similar to Theorem 2.1 and hence is omitted. ∎

3 Examples

In this section, we present an example to illustrate our results established in Section 2.

Example 1. *Consider the following impulsive partial differential equation*

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t}\left[4t^2\left(\dfrac{\partial}{\partial t}\left(u(x,t)+\dfrac{1}{4}u\left(x,t-\dfrac{1}{3}\right)\right)\right)\right]+(-8t)\left(\dfrac{\partial}{\partial t}\left(u(x,t)+\dfrac{1}{4}u\left(x,t-\dfrac{1}{3}\right)\right)\right)\\ +\dfrac{1}{(3t-1)^2}u\left(x,t-\dfrac{1}{3}\right)+\dfrac{2}{(3t-1)^2}u\left(x,t-\dfrac{1}{3}\right)=3\Delta u(x,t)+\dfrac{(54t^2+9)}{(3t-1)^2}\Delta u\left(x,t-\dfrac{1}{3}\right)+E(x,t),\quad t\neq t_k,\\ u(x,t_k^+)=\dfrac{k}{k+1}u(x,t_k)\\ u_t(x,t_k^+)=u_t(x,t_k),\quad k=1,2,\cdots,\end{array}\right\} \tag{3.1}$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$, *with the boundary condition*

$$u_x(0,t)=u_x(\pi,t)=0,\qquad (x,t)\in\partial\Omega\times\mathbb{R}_+. \tag{3.2}$$

Here $\Omega=(0,\pi)$, $N=1$, $r(t)=t^2$, $p(t)=-2t$, $a(t)=3$, $a_1(t)=\dfrac{(54t^2+9)}{(3t-1)^2}$, $j=1$, $h(t)=\dfrac{1}{4}$, $q(t)=\dfrac{1}{(3t-1)^2}$, $q_1(t)=\dfrac{2}{(3t-1)^2}$, $i=1$, $\rho(t)=\sigma(t)=\delta_1(t)=t-\frac{1}{3}$, $f(u)=u$, $\epsilon=\epsilon_1=1$, $g(u)=4u$, $E(t)=\dfrac{2(54t^2+9)}{(3t-1)^3}\cos x+\dfrac{11}{t}\cos x$, $\alpha_k=\alpha_k^*=\dfrac{k}{k+1}$, $\beta_k=\beta_k^*=1$, $t_0=1$, $t_k=2^k$. *Hence, it is easy to see that all conditions of Theorem 2.1 are satisfied. In fact* $u(x,t)=\dfrac{\cos x}{t}$ *is one such solution.*

Conclusion: In this paper, we have discussed the asymptotic behavior of impulsive neutral partial differential equations. We have obtained several new sufficient conditions for the asymptotic and oscillatory behavior of equation (1.1), together with the boundary condition (1.2), these conditions extends and complements the some of the results already existing in the previous literature [7, 16].

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