American J. of Mathematics and Sciences
Vol. 3, No -1 ,(January 2014)

ISSN No: 2250-3102

ON CENTER OF FINITELY GENERATED LOCALLY (-1,1) RINGS

K.Jayalakshmi
Assistant Professor in Mathematics,
J.N.T.University Anantapur College of Engg.
J.N.T.University Anantapur.
Anantapur.(A.P) INDIA.
jayalakshmikaramsi@gmail.com

C.Manjula
Department of Mathematics,
J.N.T.University Anantapur College of Engg.
J.N.T.University Anantapur.
Anantapur.(A.P) INDIA.
man7ju@gmail.com

ABSTRACT : A simple finitely generated locally (-1,1) ring must be an associative field.

2010 MATHEMATICS SUBJECT CLASSIFICATION : 17D20

KEY WORDS : Locally (-1,1) ring, nilpotent ideal, simple ring.

INTRODUCTION : Hentzel and smith [1] studied simple locally (-1,1) nil rings and showed that a simple locally (-1,1) nil ring of char. ≠ 2,3 must be associative. Hentzel [1] studied properties of nil potent ideals in semi simple (-1,1) rings which are nil. We concentrate mainly on the result of Hentzel [1] and prove that a simple finitely generated locally (-1,1) ring must be an associative field.
A ring is a (-1,1) ring if it is satisfies the conditions:

$$0 \equiv A(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y). \qquad \dots (1)$$

$$0 \equiv B(x,y,z) = (x,y,z) + (x,z,y). \qquad \dots (2)$$

A ring is locally (-1,1) if the subring generated by any two of its elements is (-1,1). For example, both (-1,1) rings and alternative rings are locally (-1,1). In a nonassociative ring R, we define $(x,y,z) = (xy)z - x(yz)$ and $[x,y] = xy - yx$ for all $x,y \in R$. A ring R is said to be simple if whenever A is an ideal of R then either $A = R$ or $A = 0$. By the center Z of R we mean that the set of all elements z in N such that $[z,R] = 0$ i.e., $Z = \{z \in N \mid [z,R] = 0\}$. That is C represents set of all elements which commutes with all elements in the ring and c will always means and elements taken from C. We use the following identities which hold in locally (-1,1) ring char. ≠ 2,3, which were proved by Hentzel [1]. Whereas the commutative center C is defined as $C = \{c \in R \mid [c,R] = 0\}$.

$$0 \equiv C(x,y,z) = (x,y,yz) - (x,y,z)y. \qquad \dots (3)$$

$$0 \equiv D(x,y,z,w) = (x,yz,w) + (x,wz,y) - (x,z,w)y - (x,z,y)w. \qquad \dots (4)$$

$0 \equiv E(x,y,z) = (x,y^2,z) - (x,y,yz + zy).$ … (5)

$0 \equiv F(x,y,y',z) = (x,yy' + y'y,z) - (x,y,y'z + z\,y') - (x,y',yz + zy).$ … (6)

$0 \equiv G(x,y,z) = [x,yz] + [y,zx] + [z,xy].$ … (7)

$0 \equiv H(x,y,z) = [x,[y,z]] + [y,[z,x]] + [z,[x,y]].$ … (8)

$0 \equiv I(x,y,z,w) = (xy,z,w) - (x,yz,w) + (x,y,zw) - x(y,z,w) - (x,y,z)w.$ … (9)

$0 \equiv J(x,y,z) = [x,(y,z,x)] + [x,(z,y,x)].$ … (10)

$0 \equiv K(x,y,z) = [x,(y,y,z)] + [z,(y,y,x)].$ … (11)

$0 \equiv L(x,y,z) = [x,(y,y,z)] - 3[y,(x,z,y)].$ … (12)

$0 \equiv M(x,y,z) = [xy,z] - x[y,z] - [x,z]y - 2(x,y,z) - (z,x,y).$ … (13)

$0 \equiv N(x,y,z,w) = (xy,z,w) + (x,y,[z,w]) - x(y,z,w) + (x,z,w)y.$ … (14)

$0 \equiv O(x,y,z,w) = ([x,y],z,w) - ([z,w],x,y) - [x,(y,z,w)] + [y,(x,z,w)].$ … (15)

$0 \equiv P(x,y,z,w) = [x,(y,z,w)] - [y,(z,w,x)] + [z,(w,x,y)] - [w,(x,y,z)].$ … (16)

$0 \equiv Q(x,y,c) = (x,y,c) + (y,x,c).$ … (17)

$0 \equiv R(c,x,y) = (c,x,y) - 2(y,x,c).$ … (18)

$0 \equiv \mathrm{S}(x,y,c) = 3(x,y,c) - [x,y]c + [x,yc].$ … (19)

If S is a subset of a locally (-1,1) ring R, by S^c we mean $\{x / 2^i 3^i x \in S$ for some $0 \leq i,j\}$. It is easily shown $S^c \cdot T^c \subseteq (ST)^c$ and $(S^c)^c = S^c$. We call a set S fat if $S^c = S$.

We need the following theorem proved by kleinfeld.

THEOREM 1: In any (-1,1) ring $(x,y,(x,x,y)) = (y,x,(x,x,y)) = ((x,x,y),x,y) = (x,x,y)[x,y] = 0$.

PROOF : The proof of the theorem can be found in [4, Lemmas 1,2].

LEMMA 1: $A \subseteq \{x \setminus 3^i x \in$ additive subgroup generated by the set of all (y,y,r) for all $y,r \in R\}$.

PROOF : Let $M = \{x \setminus 3^i x \in$ additive subgroup generated by the set of all (y,y,r) for all $y,r \in R\}$. $(R,R,R) \subseteq M$ by [1, Lemma 2]. To show M is an ideal, by $0 \equiv I$ it is only necessary to show $x(y,y,r) \in M$ for all x,y,r. This follows from $N(x,y,y,r) - C(x,y,r)$. ♦

If R is a locally (-1,1) ring and $k \in R$, define $T_a : R \rightarrow R$ by $rT_k = rk$ (right multiplication by k). T_k is an element of the associative ring of all endomorphism on the abelian group $(R,+)$. Let T_R = the subring of endomorphism on $(R,+)$ generated by $\{T_k \backslash k \in R\}$. Let $I = (R,R,R)^c$. I is an ideal of R and $I \subseteq \{(x,x,R) \setminus x \in R\}^c$ Lemma 1. Let T_1= the ideal of T_R generated by $\{T_k \setminus k \in I\}$. We shall now derive the following identities.

THEOREM 2: Let R be a finitely generated locally (-1,1) ring, then T_i is a nilpotent ideal of T_R.

$[R,R] \subseteq C.$ … (20)

$(C,C,R) = (C,R,C) = (R,C,C) = 0.$ … (21)

$(x,x,C) = 0.$ … (22)
$(x,y,c)c' = (x,c'y,c).$ … (23)
C is a commutative associative subring of R. … (24)
(vi) $(x,x,y)c = (x,x,cy).$ … (25)

PROOF : We prove equation (20) from equation (8).
$0 = [[R,R],R] + [[R,R],R] + [R,R],R]$
$0 = [[R,R],R]$ by char $\neq 3$ we obtain $[R,R] \subseteq C$.
(21) follows from $0 \equiv Q$ and $0 \equiv R$.
(22) follows from $0 \equiv Q$.
(23) follows from $0 = N(c',y,x,c) - Q(x,c'y,c) + c' \cdot Q(y,x,c)$ and(ii).
(24) follows from (21) and $0 \equiv M$.
(25) follows from $0 = 2D(x,c,y,x) + R(c,xy,x) - R(c,y,x) \cdot x + C(c,x,y) - B(c,x,xy) + B(c,x,y) \cdot x + 2B(x,x,y) \cdot c - 2B(c,cy,x)$.
The proof of Theorem (1) begins. ♦

LEMMA 2: (a) $(C,R,R) + (R,R,C) \subseteq C$.
(b) $(C,R,[R,R]) = ([R,R],R,C) = 0$.

PROOF : From [1, Lemma 5] we have $(C,R,R) \subseteq C$. Since $(x,y,c) = (z,y,x) - R(c,y,x)$, by char. $\neq 2$, $(R,R,C) \subseteq C$. To prove (b), from [1, Corollary 1] we have $(C,R,[R,R]) = 0$. To second part is from $([x,y],r,c) = (c,r,[x,y]) - R(c,r,[x,y])$. ♦

LEMMA 3: Let $W = (R,R,C)$ then $(R,R,W^c) \subseteq W^i$.

PROOF : This is proved by induction. Since $W \subseteq C$ by Lemma 2, $(R,R,W^1) \subseteq W^1$, and the result is true for $i=1$. We now show $(R,R,W^r) \subseteq W^r$ and $(R,R,W^s) \subseteq W^s$ implies $(R,R,W^{r+s}) \subseteq W^{r+s}$. $(R,R,W^rW^s) \subseteq (R,W^r,RW^s) + (R,W^r,W^s)R + (R,R,W^s)W^r$ by $0 \equiv D \subseteq (R,W^r,R)W^s + 0 + (R,R,W^s)W^r$ by (23) and (21) $\subseteq W^{r+s}$ by induction. This finishes the poof of Lemma 3. If $S \subseteq R$, let $(S)\#$ = ideal of R generated by S. ♦

LEMMA 4: $(W^i)\# = W^i + W^iR$.

PROOF : It is sufficient to show that $W^i + W^iR$ is an ideal of R. $(W^i + W^iR)R \subseteq W^iR + W^i \cdot R^2 - (W^i,R,R) \subseteq W^iR + (R,R,W^i)$ by $0 \equiv R \subseteq W^i + W^iR$. $R(W^i + W^iR) \subseteq RW^i + R(RW^i) \subseteq RW^i + (R,R,W^i) \subseteq W^i + W^iR$. Therefore $W^i + W^iR$ is an ideal of R. ♦

LEMMA 5: $(W^i)\# \cdot (W^j)\# \subseteq (W^{i+j})\#$.

PROOF : We do this proof in two parts. First $W^i \cdot (W^j)\# = W^i(W^j + W^jR) \subseteq W^{i+j} + W^{i+j}R$ by (21). Second $W^iR \cdot (W^j)\# \subseteq W^i \cdot R(W^j)\# + (W^i,(W^j)\#,R) \subseteq W^i(W^j)\# + W^i(W^j)\# \cdot R \subseteq$

$(W^{i+j})\#$ by the first part. ♦

LEMMA 6: If R is generated by a set of n elements G, then $W^{n+1} = 0$.

PROOF : We do this proof in three parts. First: $(C,R,R) \subseteq \sum_{g \in G} (C,g,R)$

$2(c,xy,r) = (c,xy + yx,r) + (c,[x,y],r) = (c,xy + yx,r)$ by (20) and (21) $= (c,x,yr + ry) + (c,y,xr + rx)$ by $0 \equiv F = 2(c,x,yr) + 2(c,y,xr)$ by (20) and (21).
Second: $(C,a,R)(C,a,R) = 0$.
$(C,a,R)(C,a,R) \subseteq (C,(C,R)a,R)$ by (23) $\subseteq (C,(C,a,aR),R)$ by $0 \equiv C = 0$ by Lemma 1 and (21).
Third: By $0 \equiv R$, $2W \subseteq (C,R,R)$. Thus $2^{n+1}W^{n+1} \subseteq (C,R,R)^{n+1}$. We will show $(C,R,R)^{n+1} = 0$.

$(C,R,R)^{n+1} \subseteq \sum \prod_{i=1}^{n+1} (C,x_i,R)$, where $x_i \in G$ by the first part. In each product $\prod_{i=1}^{n+1} (C,x_i,R)$ at least two of the x_i are identical as there are n+1 x_i's taken from a set G containing n elements. By the second part $\prod_{i=1}^{n+1} (C,x_i,R) = 0$.

We have shown $W^{n+1} = 0$. Let $\langle W^i \rangle = ((W^i)\#)^c$. For each I, $\langle W^i \rangle$ is an ideal of R, and from Lemma 5 we have $\langle W^j \rangle \subseteq \langle W^{i+j} \rangle$. ♦

LEMMA 7: $I^2 \subseteq \langle W^1 \rangle$.

PROOF : This proof takes four steps: (7.1),(7.2),(7.3) and (7.4).

$(a,a,x^2) = (a,ax + xa,x)$ by $0 \equiv E = 2(a,a,x)x + (a,[a,x],x)$ by $0 \equiv C \cdot 2(a,a,bc) = (a,a,bc + cb) + (a,a,bc + cb)$ by (i) and (iii). Combining these two statements gives use

$2(a,a,bc) = 2(a,a,b)c + 2(a,a,c)b + (a,[a,b],c) + (a,[a,c],b)$. … (7.1)

We now show: $[R,I] \subseteq \langle W \rangle$. … (7.2)

$3R([a,c],a,b) \in W$. By Lemma 2 we have $[R,(R,R,R)] \subseteq W$ and thus $[R,I] \subseteq W$.

$c \in I$ implies $(a,a,c) \in \langle W \rangle$ … (7.3)

$(a,a,c) = [c,a]a - [ca,a] + M(c,a,a) + B(a,c,a) \in \langle W \rangle$. … (7.4)

Let $c \in I$. By (7.1) $2(a,a,bc) = 2(a,a,b)c + 2(a,a,c)b + (a,[a,b],c) + (a,[a,c],b)$; $(a,[a,b],c)$ and $(a,[a,c],b)$ are in $\langle W \rangle$ by (7.3). The remaining term $2(a,a,b)c$ must also be in W. We have shown $(a,a,b)I \subseteq \langle W \rangle$ and thus $I^2 \subseteq W$. ♦

LEMMA 8: $(I,I,W^i) \subseteq \langle W^{i+1} \rangle$.

PROOF : The proof of Lemma 8 takes four steps.

$[(a,a,b),bc] = [(a,a,b)c,b] = [(a,a,cb),b] = -[(a,a,b),cb]$ … (8.1)

By $0 \equiv G$, (vi) and $0 \equiv K$. Therefore $[(a,a,b),bc] = 0$.

$((a,a,b),b,c) = 0$. … (8.2)

$3((a,a,b),b,c) = [(a,a,b),b]c - [(a,a,b),bc] + S((a,a,b),b,c) = 0$ by $0 \equiv J$ and (8.1).

$(I,I,W^i) \subseteq \langle W^{i+1} \rangle$. … (8.3)

If $c \in I$ $((a,a,b),c,z) = -(a,a,c),b,c)$ by (8.2) $\in (\langle W \rangle,b,c)$ by (7.3). Hence $((a,a,b),c,W^i) \subseteq (\langle W \rangle,R,\langle W^i \rangle) \subseteq \langle W^{i+1} \rangle$. We have now shown $(I,I,W^i) \subseteq \langle W^{i+1} \rangle$; this completes the poof of Lemma (8). ♦

LEMMA 9: $\langle W^i \rangle I \cdot I \subseteq W^{i+1}$.

PROOF : $(W^i)\# I \cdot I \subseteq (W^i)\# \cdot I^2 + ((W^i)\#,I,I)$

$\subseteq + (W^i,I,I) + (W^iR,I,I)$ by Lemmas (4),(5) and (7).

$\subseteq \langle W^{i+1} \rangle + W^i(R,I,I) + (W^i,I,I)R + (W^i,R,[I,I])$

By $0 \equiv N \subseteq \langle W^{i+1} \rangle$ by Lemmas 5,7 and 8. ♦

LEMMA 10: If R has n generators, then $T_1^{2n+2} = 0$.

PROOF : Let $I_0 = R$ and define inductively $I_{i+1} = I_i \cdot I_1$. It is easy to show I_i is a right ideal for each I and $(T_1)^I \subseteq I_i$. By Lemma 8, $I_{2i} \subseteq \langle W^i \rangle$. This means $R(T_1)^{2n+2} \subseteq I_{2n+2} \subseteq \langle W^{n+1} \rangle = 0$.

We have finished the proof of Theorem 1. ♦

LEMMA 11: In a finitely generated locally (-1,1) ring R, $x \in (x(a,b,c)T_R)^c$ implies $x = 0$.

This means that if P is the right ideal generated by $x(a,b,c)$ which has all right multiples of $x(a,b,c)$, but not necessarily $x(a,b,c)$ as R might not have an identity, this right ideal is always a proper right ideal, and even if you enlarge it to P^c, it still is a proper right ideal.

PROOF : If $2^i 3^i x = x(a,b,c)\tau$ for some $\tau \in T_R$ then $2^i 3^i x = xT_{(a,b,c)}\tau$ and iterating $(2^i 3^i)^n x = x(T_{(a,b,c)}\tau)^n = 0$ for suitable index $n > 0$ as $T_{(a,b,c)}\tau \in$ the ideal T_1 which is nilpotent. Therefore $x = 0$. ♦

LEMMA 12: Suppose R is not necessarily generated. Here also $x \in (x(a,b,c)T_R)^c$ implies $x = 0$.

PROOF : If $x \in (x(a,b,c)T_R)^c$ then $2^i 3^i x = xT_{(a,b,c)}\tau$ for some $\tau \in T_R$. τ is a combination of sums and products of a finite number of elements of the form $T_r : r \in R$. Let $R\#$ be the subring generated by a,b,c,x and the elements of which τ was made. In $R\#$ $x \in (x(a,b,c)T_{R\#})^c$ so $x = 0$. ♦

LEMMA 13: If R has no proper fat right ideals then R is associative.

PROOF : I is a fat right ideal (actually, a fat two-sided). Thus (i) $I = 0$ and R is associative or
(ii) $I = R$. In this case $R(R,R,R) \cdot R = 0$ by Lemma 12; so $R = 0$. ♦

LEMMA 14: If R has no proper ideals then R has no proper fat right ideals.

PROOF : Assume R has no proper ideals and that P is a proper fat right ideal of R. If $z \in P$ then $(R,R,z) \subseteq P$ since $(a,b,z) = (z,b,a)$ by $0 \equiv R$.

We continue by letting $A_1 = z$,
$A_2 = (R,R,A_1)$,
$A_{n+1} = (R,R,A_n)$.

Let $A = \cup A_i$. Now $A \subseteq Z$ and $A \subseteq P$; $A + AR \subseteq P$ and $A + AR$ is a 2 ideal. Thus $A = 0$. So $P \cap Z = 0$. Now $[P^2,R] \subseteq Z$ and $[P^2,R] \subseteq [PR,P] \subseteq P$ by $0 \equiv G$ and (20); therefore $[P^2,R] = 0$. Thus $p^2 \in P \cap Z$ so $p^2 = 0$. Furthermore $(R,P,P) \subseteq (P,R,R) = 0$; so $RP \cdot P = 0$. Let $P_1 = P + RP + (R,R,P)$. P_1 is a right ideal since $(R,R,P)R \subseteq (R,R,R)P + (R,RR,P) + (R,PR,R)$ by $0 \equiv D \subseteq RP + (R,R,P) \subseteq P_1$. We will show $P_1^c \neq R$. $P_1 P \subseteq P^2 + (RP) + (R,R,P)P$
$\subseteq 0 + 0 + (R,R,P)R + (R,R,P^2) + (R,P.RP)$ by $0 \equiv D$
$\subseteq (R,P,RP) \subseteq (R,P,PR) + (R,P,[R,P]) \subseteq (P,R,[R,P])$
$\subseteq P$ by (i) and $0 \equiv Q$.

Now $P_1^c P^c \subseteq (P_1 P)^c \subseteq P$. If $P_1^c = R$ then $RP \subseteq P$ and P is a two-sided, impossible. Thus $P_1^c \neq R$. Let us repeat this construction.

$P_1 = (P + RP + (R,R,P))^c$,
$P_2 = (P_1 + RP_1 + (R,R,P_1))^c$,
$P_3 = (P_n + RP_n + (R,R,Pn))^c$.

$P_i \neq R$ for all I, so $P_i^2 = 0$. Since $\cup P_i$ is a two-sided, we have $R^2 = 0$; this means $RP \subseteq P$. Therefore P is a two-ideal, contradiction. ♦

THEOREM 2: If R is a simple locally (-1,1) ring then is an associative field.

PROOF : If R has no proper ideals, by Lemma 14 R has no proper fat right ideals and by Lemma 13 R is associative. The center of R is 0 or a field. $[R,R] \subseteq$ center. This implies $[x,y]^3 = 0$; hence $[x,y] = 0$. R must be commutative. A simple associative commutative ring is a field. So R is a field. ♦

REFERENCES

1. Hentzel.I.R. and Smith.H.F, " Simple locally (-1,1) nil rings", Journal of algebra 262- 272.Vol.101.No.1.June 1986.

2. Kleinfeld.E, "A Generalization of strongly (-1,1) rings", Journal of algebra 119, 218-225(1988).
3. Suvarna.K and Jayalakshmi.K, "A Result on prime (-1,1) rings", proceedings of international conference on advances in mathematical and computational methods (AMCM-2011),Vol.1, 44-46.
4. Subhashini.K, "Simplicity on Accessible and (-1,1) rings", Int.J. Contemp.Math.Sciences, Vol.7,2012,no.48,2377-2381.
5. Subhashini.K, "Simple (1,0) rings", International Mathematical Forum,Vol.7,2012,no.48, 2377-2381.